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Superparticle and superstring in $AdS_3 \times S^3$ Ramond-Ramond background in light-cone gauge

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Abstract

We discuss superparticle and superstring dynamics in $AdS_3 \times S^3$ supported by R-R 3-form background using light-cone gauge approach. Starting with the superalgebra $psu(1,1|2) \oplus \widetilde{psu}(1,1|2)$ representing the basic symmetry of this background we find the light-cone superparticle Hamiltonian. We determine the harmonic decomposition of light-cone superfield describing fluctuations of type IIB supergravity fields expanded near $AdS_3 \times S^3$ background and thus the corresponding Kaluza-Klein spectrum. We fix the fermionic and bosonic light-cone gauges in the covariant Green-Schwarz $AdS_3 \times S^3$ superstring action and find the corresponding light-cone string Hamiltonian. We also obtain a realization of the generators of $psu(1,1|2) \oplus \widetilde{psu}(1,1|2)$ in terms of the superstring 2-d fields in the light-cone gauge.

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1 Introduction

Understanding how to quantize superstrings in Ramond-Ramond backgrounds is of topical interest, in particular, in connection with string theory – gauge theory duality [1, 2]. The basic example of type IIB Green-Schwarz string in $AdS_5 \times S^5$ with R-R 5-form background [2] was studied, e.g., in [3–10]. One may hope that a progress towards understanding the spectrum of this theory may be achieved by using a light-cone gauge approach recently developed in [19, 20] (for an alternative covariant approach see [11]). To get a better understanding of this light-cone approach it may be useful to consider first a similar but somewhat simpler string model.

An obvious candidate for such simpler model is type IIB string in $AdS_3 \times S^3 \times T^4$ with R-R 3-form background.[1] The $AdS_3 \times S^3 \times T^4$ with NS-NS 3-form background represents the near-horizon limit of NS 5-brane – fundamental string configuration [12] and a fundamental superstring probe in it may be described by the standard $SL(2,R) \times SU(2)$ WZW model in the NSR formulation. However, the superstring propagation in S-dual R-R background which is the near-horizon limit of D5–D1 system [13] cannot be studied directly in the usual NSR formalism. The explicit form of the covariant GS string action in this R-R background was found in [14–16] by applying the same supercoset method which was used in the $AdS_5 \times S^5$ string case in [3]. An alternative "hybrid" approach to quantization of superstring in $AdS_3 \times S^3$ R-R background was developed in [17] (see also [18]).

In this paper we shall discuss several aspects of superstring dynamics in the $AdS_3 \times S^3$ R-R background in the light-cone approach developed for the $AdS_5 \times S^5$ case in [19, 20]. Since the simplest limiting case of superstring is superparticle, we also consider in some detail the light-cone superparticle theory in $AdS_3 \times S^3$, following closely the treatment of the $AdS_5 \times S^5$ case in [21]. First quantization of superparticle determines the spectrum of fluctuations of type IIB supergravity in $AdS_3 \times S^3 \times T^4$ (found directly in component form in [22]) and thus also the "ground state" spectrum of the corresponding string theory. In the treatment of the superstring theory our starting point will be the covariant GS action (see [14–16]) where we shall fix the light-cone-type fermionic (κ-symmetry) and bosonic (2-d diffeomorphism) gauges and derive the light-cone Hamiltonian along the lines of the phase space approach of [20].

The paper is organized as follows.

In Section 2 we review the structure of the underlying symmetry superalgebra of the type IIB superstring theory in $AdS_3 \times S^3$ R-R background – $psu(1,1|2) \oplus \widetilde{psu}(1,1|2)$ [23] and present its (anti)commutation relations in a light-cone basis.

In Section 3 we consider superparticle dynamics in $AdS_3 \times S^3$. We find the light-cone superparticle Hamiltonian and a realization of the generators of $psu(1,1|2) \oplus \widetilde{psu}(1,1|2)$ on phase space of (first-quantized) superparticle.

In Section 4 we develop a manifestly supersymmetric light-cone gauge formulation of type IIB supergravity on $AdS_3 \times S^3$ background. The quadratic term in the action for fluctuation fields is written in terms of a single unconstrained scalar light-cone superfield, allowing us to treat all the component fields on an equal footing. We also present a

[1] In what follows we shall ignore the trivial T^4 factor.

superfield version of S^3 harmonic decomposition and find the corresponding K-K spectra of the supergravity modes propagating in AdS_3.

In Section 5 we find the κ-symmetry light-cone gauge fixed form of the superstring action in $AdS_3 \times S^3$. We give the superstring Lagrangian both in the "Wess-Zumino" and "Killing" parametrizations of the basic coset superspace $[PSU(1,1|2) \times \widetilde{PSU}(1,1|2)]/[SO(2,1) \times SO(3)]$ on which the superstring is propagating. We also discuss a reformulation of the resulting superstring action in terms of 2-d Dirac world-sheet fermions.

Section 6 is devoted to the light-cone phase space approach to superstring theory. We fix the analog of the GGRT bosonic light-cone gauge and derive the phase space analog of the superstring Lagrangian of Section 5 and the corresponding light-cone gauge Hamiltonian.

In Section 7 we obtain a realization of the generators of the symmetry superalgebra $psu(1,1|2) \oplus \widetilde{psu}(1,1|2)$ as Noether charges expressed in terms of the 2-d fields which are the coordinates of the $AdS_3 \times S^3$ superstring in the light cone gauge.

Some technical details are collected in five Appendices. In Appendix A we summarize our notation and definitions used in this paper and give some relations relevant for a coset description of S^3. In Appendix B we describe correspondence between the "covariant" and "light-cone" forms of the $psu(1,1|2) \oplus \widetilde{psu}(1,1|2)$ superalgebra. In Appendix C we explain the construction of Poincaré supercharges in the case of superparticle. In Appendix D we give some details of computation of the spectrum of type IIB supergravity fluctuations in $AdS_3 \times S^3$. In appendix E we present the expressions for the supercoset Cartan 1-forms which are the basic elements in the construction of the GS superstring action, and describe our procedure of fixing the fermionic light-cone gauge in the string action.

2 $psu(1,1|2) \oplus \widetilde{psu}(1,1|2)$ superalgebra

The symmetry algebra of the $AdS_3 \times S^3$ with R-R 3-form background may be represented as a direct sum of two copies of $psu(1,1|2)$ superalgebra, i.e. as $psu(1,1|2) \oplus \widetilde{psu}(1,1|2)$ superalgebra [23]. The even part of this superalgebra consists of the bosonic subalgebras $su(1,1)$, $su(2)$ and $\widetilde{su}(1,1)$, $\widetilde{su}(2)$ respectively. $su(1,1)$ and $\widetilde{su}(1,1)$ combine into $so(2,2)$ algebra while $su(2)$ and $\widetilde{su}(2)$ form $so(4)$ algebra. These $so(2,2)$ and $so(4)$ algebras are the isometry algebras of the AdS_3 and S^3 factors respectively. The odd part of the superalgebra consists of 16 supercharges which correspond to the 16 Killing spinors of $AdS_3 \times S^3$ geometry.

The superalgebra $psu(1,1|2) \oplus \widetilde{psu}(1,1|2)$ will play the central role in our constructions. Let us review its commutation relations in the two forms ("covariant" and "light-cone") we are going to use. In $su(1,1) \oplus su(2)$ covariant basis the $psu(1,1|2)$ superalgebra has the following generators: $m^\alpha{}_\beta$ and $m^i{}_j$ which are generators of $su(1,1)$ and $su(2)$ and 8 supercharges q^α_i, q^i_α ($\alpha, \beta = 1,2;\ i,j = 1,2$). Their (anti)commutation relations have the following well known form

$$[m^\alpha{}_\beta, m^\gamma{}_\delta] = \delta^\gamma_\beta m^\alpha{}_\delta - \delta^\alpha_\delta m^\gamma{}_\beta\,, \qquad [m^i{}_j, m^k{}_n] = \delta^k_j m^i{}_n - \delta^i_n m^k{}_j\,, \tag{2.1}$$

$$[m^\alpha{}_\beta, q^k_\gamma] = -\delta^\alpha_\gamma q^k_\beta + \frac{1}{2}\delta^\alpha_\beta q^k_\gamma\,, \qquad [m^i{}_j, q^k_\alpha] = \delta^k_j q^i_\alpha - \frac{1}{2}\delta^i_j q^k_\alpha\,, \tag{2.2}$$

$$[m^i{}_j, q^\alpha_k] = -\delta^i_k q^\alpha_j + \frac{1}{2}\delta^i_j q^\alpha_k\,, \qquad [m^\alpha{}_\beta, q^\gamma_k] = \delta^\gamma_\beta q^\alpha_k - \frac{1}{2}\delta^\alpha_\beta q^\gamma_k\,, \tag{2.3}$$

$$\{q^i_\alpha, q^\beta_j\} = a(\delta^i_j m^\beta{}_\alpha + \delta^\beta_\alpha m^i{}_j)\,, \qquad a^2 = -1\,. \tag{2.4}$$

We assume the following Hermitean conjugation rules

$$(m^\alpha{}_\beta)^\dagger = -m^\alpha{}_\beta\,, \qquad (m^i{}_j)^\dagger = m^j{}_i\,, \qquad (q^\alpha_i)^\dagger = \epsilon^{\alpha\beta} q^i_\beta\,, \qquad (q^i_\alpha)^\dagger = q^\beta_i \epsilon_{\beta\alpha}\,, \tag{2.5}$$

where $\epsilon^{\alpha\beta}$ is the Levi-Civita tensor: $\epsilon_{12} = \epsilon^{12} = 1$. The $\widetilde{psu}(1,1|2)$ superalgebra has the same commutation relations but with the constant a in (2.4) replaced by $\tilde{a}$ ($\tilde{a}^2 = -1$) such that its sign is opposite to that of a, i.e. $a\tilde{a} = 1$.

It will be useful to decompose the generators according to their light-cone $SO(1,1)$ group transformation properties (we shall call this "light-cone basis"). In the light-cone basis the generators of $psu(1,1|2) \oplus \widetilde{psu}(1,1|2)$ include translations $P^\pm$, conformal boosts $K^\pm$, Lorentz rotation J^{+-}, dilatation D, R-symmetry generators of $su(2)$ and $\widetilde{su}(2)$ $J^i{}_j$ and $\tilde{J}^i{}_j$, Poincaré algebra supercharges $Q^{\pm i}$ and conformal algebra supercharges $S^{\pm i}$. To simplify the notation here we use the same type of indices for $su(2)$ and $\widetilde{su}(2)$. The Hermiteant conjugation rules are

$$(P^\pm)^\dagger = P^\pm\,, \qquad (K^\pm)^\dagger = K^\pm\,, \qquad (Q^{\pm i})^\dagger = Q^\pm_i\,, \qquad (S^{\pm i})^\dagger = S^\pm_i\,, \tag{2.6}$$

$$(J^{+-})^\dagger = -J^{+-}\,, \quad D^\dagger = -D\,, \quad J^{i\dagger}_j = J^j{}_i\,, \qquad \tilde{J}^{i\dagger}_j = \tilde{J}^j{}_i\,. \tag{2.7}$$

The anti(commutation) relations then include (their derivation from the above relations is explained in Appendix B)

$$[P^\pm, K^\mp] = D \mp J^{+-}\,, \tag{2.8}$$

$$[D, P^\pm] = -P^\pm\,, \quad [D, K^\pm] = K^\pm\,, \quad [J^{+-}, P^\pm] = \pm P^\pm\,, \quad [J^{+-}, K^\pm] = \pm K^\pm\,, \tag{2.9}$$

$$[D, Q^\pm_i] = -\frac{1}{2}Q^\pm_i\,, \quad [D, S^\pm_i] = \frac{1}{2}S^\pm_i\,, \quad [J^{+-}, Q^\pm_i] = \pm\frac{1}{2}Q^\pm_i\,, \quad [J^{+-}, S^\pm_i] = \pm\frac{1}{2}S^\pm_i\,, \tag{2.10}$$

$$[S^\mp_i, P^\pm] = Q^\pm_i\,, \quad [Q^{\mp i}, K^\pm] = S^{\pm i}\,, \quad \{Q^{\pm i}, Q^\pm_j\} = \pm P^\pm \delta^i_j\,, \quad \{S^{\pm i}, S^\pm_j\} = \pm K^\pm \delta^i_j\,, \tag{2.11}$$

$$\{Q^{+i}, S^-_j\} = \frac{1}{2}(J^{+-} - D)\delta^i_j - \tilde{J}^i{}_j\,, \qquad \{Q^{-i}, S^+_j\} = \frac{1}{2}(J^{+-} + D)\delta^i_j + J^i{}_j\,, \tag{2.12}$$

plus Hermitean conjugations of the above ones. The remaining relations can be summarized as follows. The supercharges Q^-_i, Q^{-i}, S^{+i}, S^+_i transform in the (anti)fundamental representations of $su(2)$ – they are rotated only by $J^i{}_j$, i.e.

$$[J^i{}_j, Q^{-k}] = \delta^k_j Q^{-i} - \frac{1}{2}\delta^i_j Q^{-k}\,, \qquad [J^i{}_j, Q^-_k] = -\delta^i_k Q^-_j + \frac{1}{2}\delta^i_j Q^-_k\,, \tag{2.13}$$

and the same for S^+_i, S^{+i}. The remaining supercharges Q^{+i}, Q^+_i, S^{-i}, S^-_i transform in the (anti)fundamental representations of $\widetilde{su}(2)$ – they are rotated only by $\tilde{J}^i{}_j$, i.e.

$$[\tilde{J}^i{}_j, Q^{+k}] = \delta^k_j Q^{+i} - \frac{1}{2}\delta^i_j Q^{+k}\,, \qquad [\tilde{J}^i{}_j, Q^+_k] = -\delta^i_k Q^+_j + \frac{1}{2}\delta^i_j Q^+_k\,, \tag{2.14}$$

and the same for S^{-i}, S^-_i. The generators $J^i{}_j$, $\tilde{J}^i{}_j$ satisfy the standard relations

$$[J^i{}_j, J^k{}_n] = \delta^k_j J^i{}_n - \delta^i_n J^k{}_j\,, \qquad [\tilde{J}^i{}_j, \tilde{J}^k{}_n] = \delta^k_j \tilde{J}^i{}_n - \delta^i_n \tilde{J}^k{}_j\,. \tag{2.15}$$

3 Superparticle dynamics in $AdS_3 \times S^3$ backrgound

Before discussing superstring it is instructive to consider first a superparticle propagating in $AdS_3 \times S^3$ space. The covariant Brink-Schwarz κ-symmetric action for a superparticle in $AdS_3 \times S^3$ can be obtained, e.g., from the superstring action of [14–16] by taking the zero slope limit $\alpha' \to 0$. By applying the light-cone gauge fixing procedure (see [20] and below) one could then obtain the superparticle light-cone gauge fixed action. One the other hand, there is a method [24] which reduces the problem of constructing a new (light-cone gauge) dynamical system to the problem of finding a new solution of the commutation relations of the defining symmetry algebra (in our case $psu(1,1|2) \oplus \widetilde{psu}(1,1|2)$). This method of Dirac was applied to the case of superparticle in $AdS_5 \times S^5$ in [21] (see also [25]) and here we would like to demonstrate how it works for the superparticle in $AdS_3 \times S^3$. Quantization of superparticle determines the quadratic part of the action of type IIB supergravity expanded near $AdS_3 \times S^3$ background.

In the light-cone formalism the generators of the $psu(1,1|2) \oplus \widetilde{psu}(1,1|2)$ superalgebra can be split into the two groups:

$$P^+, K^+,\, Q^{+i},\, Q^+_i,\, S^{+i},\, S^+_i,\, D,\, J^{+-},\, J^i{}_j,\, \tilde{J}^i{}_j\,, \tag{3.1}$$

which we shall refer to as kinematical generators, and

$$P^-,\;\, K^-\,, Q^{-i},\, Q^-_i\,, S^{-i},\, S^-_i\;\,, \tag{3.2}$$

which we shall refer to as dynamical generators. The kinematical generators have positive or zero J^{+-} (Lorentz) charges, while the dynamical generators have negative J^{+-} charges. It turns out that in the superfield realization the kinematical generators taken at $x^+ = 0$ are quadratic in the physical fields,[2] while the dynamical generators receive higher-order interaction-dependent corrections. The first step is to find a free (quadratic) superfield representation for the generators of $psu(1,1|2) \oplus \widetilde{psu}(1,1|2)$. The generators we obtain below we will be used for the description of IIB supergravity in $AdS_3 \times S^3$ background.

Let us explain step by step how the method of [24] works in the present case: First, we introduce a light-cone superspace on which we are going to realize the generators of our superalgebra. The superspace coordinates include the position coordinates $x^\pm$, z of

[2] In general, they have the structure $G = G_1 + x^+ G_2 + (x^+)^2 G_3$ where G_1 is quadratic but G_2, G_3 contain higher order terms in second-quantized fields.

AdS_3, a unit vector u^M representing S^3, and the Grassmann coordinates θ^i, η^i. In this parametrization the metric of $AdS_3 \times S^3$ is ($M = 1, 2, 3, 4$)

$$ds^2 = \frac{1}{z^2}(2dx^+dx^- + dz^2) + du^M du^M \,, \qquad u^M u^M = 1 \tag{3.3}$$

In formulating our results we shall trade the bosonic coordinate x^- and the Grassmann coordinates θ^i, η^i for the bosonic momentum p^+ and the Grassmann momenta λ_i, ϑ_i.

Let us start with the kinematical generators and consider them on the surface of the initial data $x^+ = 0$. The kinematical generators which have positive J^{+-}-charge are fixed to be

$$P^+ = p^+ \,, \qquad K^+ = \frac{1}{2} z^2 p^+ \,, \tag{3.4}$$

$$Q_i^+ = \lambda_i \,, \quad Q^{+i} = p^+\theta^i \qquad S_i^+ = \frac{1}{\sqrt{2}} z\vartheta_i \,, \qquad S^{+i} = \frac{1}{\sqrt{2}} z p^+ \eta^i \,, \tag{3.5}$$

where the coordinates θ^i, η^i and their momenta λ_i, ϑ_i satisfy the canonical anticommutation relations

$$\{\lambda_i, \theta^j\} = \delta_i^j \,, \qquad \{\vartheta_i, \eta^j\} = \delta_i^j \tag{3.6}$$

Let us note that in the language of an action based on a supercoset construction the above parametrization of the kinematical generators corresponds to special choices of (i) coset representative and (ii) light-cone gauges for 1-d diffeomorphism symmetry and κ-symmetry. In fact, these choices may be motivated by a simple form of the resulting generators.

Once the above generators are chosen, the remaining kinematical generators which have zero J^{+-}-charge are fixed by the commutation relations of the superalgebra

$$J^{+-} = \partial_{p^+} p^+ - \frac{1}{2}\theta\lambda - \frac{1}{2}\eta\vartheta + 1 \,, \quad D = -\partial_{p^+} p^+ + z\partial_z + \frac{1}{2}\theta\lambda + \frac{1}{2}\eta\vartheta - \frac{1}{2} \,, \tag{3.7}$$

$$J^i{}_j = l^i{}_j + \eta^i\vartheta_j - \frac{1}{2}\delta^i_j \eta\vartheta \,, \qquad \tilde{J}^i{}_j = \tilde{l}^i{}_j + \theta^i\lambda_j - \frac{1}{2}\delta^i_j\theta\lambda \,, \tag{3.8}$$

where $\partial_{p^+} \equiv \partial/\partial p^+$, $\partial_z \equiv \partial/\partial z$. The orbital parts $l^i{}_j$ and $\tilde{l}^i{}_j$ of the angular momenta $J^i{}_j$ and $\tilde{J}^i{}_j$ are given by

$$l^i{}_j = \frac{1}{4}(\sigma^{MN})^i{}_j l^{MN} \,, \qquad \tilde{l}^i{}_j = \frac{1}{4}(\bar{\sigma}^{MN})^i{}_j l^{MN} \,, \tag{3.9}$$

where the $so(4)$ orbital momentum l^{MN} can be chosen as[3]

$$l^{MN} = u^M \hat{\partial}^N - u^N \hat{\partial}^M \,. \tag{3.10}$$

Here $\hat{\partial}^M$ is covariant tangent derivative on S^3 which is by fixed by the constraint $u^M\hat{\partial}^M = 0$ and by the commutation relations

[3]Note that the concrete parametrization of the S^3 part is not very important to us as in the case of the superparticle all the generators are expressed in terms of the orbital part of the angular momentum.

$$[\hat{\partial}^M, u^N] = v^{MN}\,, \qquad [\hat{\partial}^M, \hat{\partial}^N] = u^M \hat{\partial}^N - u^N \hat{\partial}^N, \qquad v^{MN} \equiv \delta^{MN} - u^M u^N\,. \quad (3.11)$$

The operator $l^i{}_j$ satisfies the following basic relation

$$l^i{}_k l^k{}_j = \frac{1}{2} l^2 \delta^i_j + l^i{}_j\,, \quad (3.12)$$

where $l^2 \equiv l^i{}_j l^j{}_i$. The same relation is true for $\tilde{l}^i{}_j$. The Hermitean conjugation rules are

$$\lambda_i^\dagger = p^+ \theta^i\,, \quad \theta^{i\dagger} = \frac{\lambda_i}{p^+}\,, \quad \vartheta_i^\dagger = p^+ \eta^i\,, \quad \eta^{i\dagger} = \frac{\vartheta_i}{p^+}\,, \quad (\partial_{p^+} p^+)^\dagger = -\partial_{p^+} p^+ + \theta\lambda + \eta\vartheta - 2. \quad (3.13)$$

Once the all the kinematical generators are fixed, the dynamical generators are found from the commutation relations of the basic superalgebra (for details see Appendix C)

$$P^- = \frac{1}{2p^+}(\partial_z^2 - \frac{1}{z^2} A)\,, \quad (3.14)$$

$$Q^-_i = \frac{1}{\sqrt{2p^+}} \Big(-\vartheta_i \partial_z - \frac{1}{z}(\eta\vartheta)\vartheta_i + \frac{1}{2z}\vartheta_i + \frac{2}{z}(\vartheta l)_i \Big)\,, \quad (3.15)$$

$$Q^{-i} = \frac{1}{\sqrt{2}} \Big(\eta^i \partial_z - \frac{1}{z}\eta^i(\eta\vartheta) + \frac{1}{2z}\eta^i + \frac{2}{z}(l\eta)^i \Big)\,, \quad (3.16)$$

$$K^- = -\bar{S}\frac{1}{p^+} S - \frac{1}{2p^+}(\tilde{l}^2 + 2\lambda\tilde{l}\theta)\,, \quad (3.17)$$

$$S^{-i} = \theta^i S - (\tilde{l}\theta)^i\,, \qquad S^-_i = \lambda_i \bar{S}\frac{1}{p^+} - \frac{1}{p^+}(\lambda\tilde{l})_i\,, \quad (3.18)$$

where the operators A, S and $\bar{S}$ are defined by

$$A \equiv X - \tfrac{1}{4}\,, \qquad X \equiv 2l^2 + 4\vartheta l \eta + (\eta\vartheta - 1)^2\,, \quad (3.19)$$

$$S \equiv -\partial_{p^+} p^+ + \tfrac{1}{2} z\partial_z + \theta\lambda + \tfrac{1}{2}\eta\vartheta - \tfrac{3}{4}\,, \qquad \bar{S} \equiv \partial_{p^+} p^+ - \tfrac{1}{2} z\partial_z - \tfrac{1}{2}\eta\vartheta + \tfrac{3}{4} \quad (3.20)$$

and we used the notation

$$(\vartheta l)_i \equiv \vartheta_j l^j{}_i\,, \quad (l\eta)^i \equiv l^i{}_j \eta^j\,, \quad (\lambda\tilde{l})_i \equiv \lambda_j \tilde{l}^j{}_i\,, \quad (\tilde{l}\theta)^i \equiv \tilde{l}^i{}_j \eta^j\,, \quad (\vartheta l \eta) \equiv \vartheta_i l^i{}_j \eta^j\,, \quad (3.21)$$

$$l^2 \equiv l^i{}_j l^j{}_i\,, \quad \tilde{l}^2 \equiv \tilde{l}^i{}_j \tilde{l}^j{}_i\,, \qquad \eta\vartheta \equiv \eta^i \vartheta_i\,, \qquad \theta\lambda \equiv \theta^i \lambda_i \quad (3.22)$$

In the light-cone approach the operator P^- plays the role of the (minus) Hamiltonian of the superparticle. The expressions for the supercharges can be rewritten as follows

$$Q^-_i = -\frac{1}{\sqrt{2p^+}} \Big(\vartheta_i \partial_z + \frac{1}{2z}[\vartheta_i, A] \Big)\,, \qquad Q^{-i} = \frac{1}{\sqrt{2}} \Big(\eta^i \partial_z + \frac{1}{2z}[\eta^i, A] \Big)\,. \quad (3.23)$$

As in [26, 27] we shall call A in (3.19) the AdS mass operator. This operator satisfies the following basic relation

$$\{[\eta^i, A], [\vartheta_j, A]\} + 2[\eta^i, A]\vartheta_j + 2[\vartheta_j, A]\eta^i = -4A\delta^i_j\,, \quad (3.24)$$

which is useful in checking that $\{Q^-_i, Q^{-j}\} = -\delta^j_i P^-$. Let us note that A is equal to zero only for massless representations which can be realized as irreducible representations of the conformal algebra [27, 28], i.e. of $so(3,2)$ in the case of AdS_3.[4] Below in Section 4.2

[4] The values of this operator for various fields are discussed in [29].

we shall demonstrate that A is not equal to zero for the whole spectrum of the $S^3 \times T^4$ compactification of type IIB supergravity to AdS_3.

The generators given above were defined on the initial data surface $x^+ = 0$. In general, they have the structure $G = G(x^+, \mathcal{X}(x^+))$ where $\mathcal{X}$ stands for all of the dynamical variables. Let us use the notation

$$G|_{x^+=0} \equiv G(0, \mathcal{X}(x^+)) \ . \tag{3.25}$$

The generators $G|_{x^+=0}$ can be obtained from the above expressions by expressing the dynamical variables $\mathcal{X}$ in terms of light-cone time variable x^+ using the Hamiltonian equations of motion which are postulated in our approach. The form of the generators for arbitrary x^+, i.e. $G(x^+, \mathcal{X}(x^+))$, is then found from the conservation laws for the charges

$$J^{+-} = J^+|_{x^+=0} + x^+ P^- \ , \qquad D = D^+|_{x^+=0} + x^+ P^- \ , \tag{3.26}$$

$$K^+ = K^+|_{x^+=0} + x^+(D|_{x^+=0} + J^{+-}|_{x^+=0}) + x^{+2} P^- \ , \tag{3.27}$$

$$S_i^+ = S_i^+|_{x^+=0} - \mathrm{i} x^+ Q_i^- \ , \qquad S^{+i} = S^{+i}|_{x^+=0} + \mathrm{i} x^+ Q^{-i} \ . \tag{3.28}$$

The remaining generators do not have explicit dependence on x^+, i.e. they have the structure $G(x^+, \mathcal{X}(x^+)) = G(0, \mathcal{X}(x^+))$.

4 Light-cone gauge superfield formulation of type IIB supergravity on $AdS_3 \times S^3$

In this Section we shall present the light-cone gauge superfield description of type IIB supergravity on $AdS_3 \times S^3$ backround, implied by the quantization of the superparticle described in the previous Section. Linearized equations of motion for fluctuations of supergravity fields in $AdS_3 \times S^3 \times K3$ background and the corresponding spectrum were found in component form [22]. We shall use instead the light-cone superfield approach.

This analysis can be viewed as a step towards understanding the spectrum of string theory in $AdS_3 \times S^3$. As is well known in the case of (super)strings in flat space, reproducing the correct spectrum of the massless modes plays an important role in determining a consistent quantization scheme. The $AdS_3 \times S^3$ spectrum we shall find below should be a useful guiding principle in quantising superstrings in this space. In particular, the operator ordering and renormalization scheme should be chosen so that the ground state of the superstring theory in $AdS_3 \times S^3$ (with R-R 3-form background)[5] will have the spectrum described below.

Finding even the quadratic part of the action for fluctuations of the supergravity fields in a curved background is a complicated problem. There are two ways of determining spectra of compactifications of the type II supergravity. The first one uses oscillator construction [30]. The second one is based on the analysis of equations of motion [22, 31]. In our construction of the spectrum we shall follow the second approach. A new element which substantially simplifies the analysis is the use of the light-cone superfield formulation.

[5] The selection of R-R as opposed to NS-NS background is pre-determined by our choice of the basic superalgebra in Section 2.

4.1 Quadratic light-cone superfield action

We could in principle use the covariant superfield description of type IIB supergavity [32], starting with linearized expansion of superfileds, imposing light-cone gauge on fluctuations and then solving the constraints to eliminate non-physical degrees of freedom in terms of physical ones. That would be quite tedious. The light-cone gauge method provides a self-contained approach which does not rely upon existence of a covariant description and which gives a much shorter way to arrive to final results. The key idea is that, as in flat space [33], the superparticle supercharges found in the previous Section provide realization of the generators of the basic $psu(1,1|2)\oplus\widetilde{psu}(1,1|2)$ superalgebra in terms of the differential operators acting on the scalar supergravity superfield $\Phi(x^\pm,z,u,\theta,\eta)$. It is convenient to Fourier transform to the momentum space for all of the coordinates except the radial AdS_3 coordinate z and S^3 directions u^M. This means using p^+, λ_i, ϑ_i instead of x^-, θ^i, η^i (λ_i and ϑ_i are in the fundamental representations of $\widetilde{su}(2)$ and $su(2)$). Thus our basic superfield will be $\Phi(x^+,p^+,z,u,\lambda,\vartheta)$ with the following expansion in powers of the Grassmann momenta λ_i and ϑ_i

$$\begin{aligned}\Phi(x^+,p^+,z,u,\lambda,\vartheta) &= p^+\phi+\lambda_i\psi_1^i+\vartheta_i\psi_2^i+(\epsilon\lambda^2)\phi_1+\lambda_i\vartheta_j\phi_2^{ij}+(\epsilon\vartheta^2)\phi_1^*\\ &+ \frac{1}{p^+}\Big((\epsilon\lambda)^i(\epsilon\vartheta^2)\psi_1^{i*}+(\epsilon\vartheta)^i(\epsilon\lambda^2)\psi_2^{i*}\Big)-\frac{1}{p^+}(\epsilon\lambda^2)(\epsilon\vartheta^2)\phi^*\,,\end{aligned} \tag{4.1}$$

where the coefficients $\phi,\phi_1,\phi_2,\psi_1,\psi_2$ are functions of x^+, the momentum p^+ and the bosonic coordinates z,u^M. We used the notation

$$(\epsilon\lambda^2)\equiv\frac{1}{2}\epsilon^{ij}\lambda_i\lambda_j\,,\qquad\qquad (\epsilon\lambda)^i\equiv\epsilon^{ij}\lambda_j \tag{4.2}$$

and the same for ϑ. The only constraint which this superfiled is to satisfy is the reality constraint

$$\Phi(-p^+,z,u,\lambda,\vartheta)=(p^+)^2\int d^2\lambda^\dagger d^2\vartheta^\dagger\; e^{(\lambda_i\lambda_i^\dagger+\vartheta_i\vartheta_i^\dagger)/p^+}(\Phi(p^+,z,u,\lambda,\vartheta))^\dagger\,, \tag{4.3}$$

where we assume the convention $(\lambda_1\lambda_2)^\dagger=\lambda_2^\dagger\lambda_1^\dagger$. This reality constraint implies that the component fields ϕ,ϕ_n are related to ϕ^*,ϕ_n^* by the Hermitean conjugation rule for the Fourier components, i.e. $(\phi^*(-p^+))^*=\phi(p^+)$, $(\phi_n^*(-p^+))^*=\phi_n(p^+)$. Eq. (4.3) leads also to the following selfduality condition

$$\phi_2^{ij}(p^+)=-\epsilon^{ik}\epsilon^{jl}\phi_2^{kl*}(-p^+)\ . \tag{4.4}$$

The light-cone action has the following 'non-covariant form'

$$S=\int dx^+dzdp^+d^3ud^2\lambda d^2\vartheta\,\Phi(-p^+,z,u,-\lambda,-\vartheta)\,[p^+(\mathrm{i}\partial_{x^+}+P^-)]\,\Phi(p^+,z,u,\lambda,\vartheta)\,, \tag{4.5}$$

where the Hamiltonian density $(-P^-)$ is given by (3.14) and d^3u stands for the S^3 volume element, i.e. $d^4u\delta(u^Mu^M-1)$.

Transforming back to the position coordinate x^- this action can be cast into 'relativistic-invariant' form

$$S=\frac{1}{2}\int d^3xd^3u\;d^2\lambda\;d^2\vartheta\,\Phi(x,u,-\lambda,-\vartheta)\;(\Box-\frac{1}{z^2}A)\;\Phi(x,u,\lambda,\vartheta) \tag{4.6}$$

where $\Box$ is the flat D'Alembertian $\Box = 2\partial_{x^-}\partial_{x^+} + \partial_z^2$, and $d^3x \equiv dx^+dx^-dz$.

As was already mentioned above, the superparticle charges found in Section 3 give the representation of $psu(1,1|2) \oplus \widetilde{psu}(1,1|2)$ in terms of differential operators acting on the supergravity superfield Φ. We can thus write down the "superfield-theory" (or "second-quantized") realization of of $psu(1,1|2) \oplus \widetilde{psu}(1,1|2)$ generators

$$\hat{G} = \int dp^+ dz d^3u \; d^2\lambda \; d^2\vartheta \;\; p^+\Phi(-p^+, z, u, -\lambda, -\vartheta) \; G \; \Phi(p^+, z, u, \lambda, \vartheta) \,, \tag{4.7}$$

where G indicates representation of $psu(1,1|2) \oplus \widetilde{psu}(1,1|2)$ superalgebra in terms of differential operators given in previous Section.

4.2 Harmonic decomposition of the light-cone superfield and the spectrum

The light-cone description given above provides a convenient way to analyse the harmonic decomposition of basic component fields and thus the corresponding spectra of fluctuation modes. A nice feature of this approach is that this can be done at the level of superfields, i.e. in a manifestly supersymmetric way. The action (4.6) gives the following equation of motion for the basic superfield Φ

$$(\Box - \frac{1}{z^2}A)\Phi = 0 \; . \tag{4.8}$$

To find the spectrum we are thus to decompose Φ into the eigenvectors of the AdS mass operator A defined in (3.19). Let us first make the standard harmonic decomposition (we absorb the coefficients of the expansion in the 'basic' vectors)[6]

$$\Phi = \sum_{k=0}^{\infty} \Phi_k \;, \tag{4.9}$$

where Φ_k are the $so(4)$ harmonic superfields, satisfying, by definition,

$$2l^2\Phi_k = k(k+2)\Phi_k \; . \tag{4.10}$$

We can further expand each Φ_k in power series with respect to the Grassmann momentum ϑ writing

$$\Phi_k = \sum_{\sigma=0}^{2} \Phi_{k,\sigma} \;, \tag{4.11}$$

where $\Phi_{k,\sigma}$ satisfies

$$2l^2\Phi_{k,\sigma} = k(k+2)\Phi_{k,\sigma} \,, \qquad\qquad \eta\vartheta\Phi_{k,\sigma} = (2-\sigma)\Phi_{k,\sigma} \; . \tag{4.12}$$

These equations tell us that the harmonic superfield $\Phi_{k,\sigma}$ is a polynomial of degree σ in the Grassmann momentum ϑ. From the expression for the operator X (3.19) it is then clear the superfields $\Phi_{k,0}$ are its eigenvectors

$$X\Phi_{k,0} = (k+1)^2\Phi_{k,0} \; . \tag{4.13}$$

[6]In this subsection the index k is used to indicate the Kaluza-Klein modes.

It is easy to demonstrate that $\Phi_{k,2}$ are also the eigenvectors of X with the same eigenvalues, i.e. $X\Phi_{k,2} = (k+1)^2\Phi_{k,2}$. This gives the following equations of motion determining the part of the mass spectrum corresponding to $\Phi_{k,0},\ \Phi_{k,2}$

$$\Big(\Box - \frac{(2k+1)(2k+3)}{4z^2}\Big)\Phi_{k,0} = 0\,, \qquad \Big(\Box - \frac{(2k+1)(2k+3)}{4z^2}\Big)\Phi_{k,2} = 0\,. \tag{4.14}$$

It turns out that the remaining superfields $\Phi_{k,1}$ are not eigenvectors of X. They can be decomposed, however, into the eigenvectors of this operator as follows (for details see Appendix B)

$$\Phi_{k,1} = \Phi_{k,1}^{(1)} + \Phi_{k,1}^{(2)}\ , \tag{4.15}$$

where

$$\Phi_{k,1}^{(1)} = (\vartheta_i - \frac{2}{k+2}(\vartheta l)_i)\Phi_{k,1}^i\,, \qquad k \ge 0\,; \tag{4.16}$$

$$\Phi_{k,1}^{(2)} = (\vartheta_i - \frac{2}{k}(\vartheta l)_i)\Phi_{k,1}^i\,, \qquad k > 0\,. \tag{4.17}$$

Here $\Phi_{k,1}^i$ does not depend on the Grassmann momentum ϑ but still depend on Grassmann momentum λ. Then

$$X\Phi_{k,1}^{(1)} = k^2\Phi_{k,1}^{(1)}\,, \qquad X\Phi_{k,1}^{(2)} = (k+2)^2\Phi_{k,1}^{(2)}\,, \tag{4.18}$$

and this gives the following equations of motion

$$\Big(\Box - \frac{(2k-1)(2k+1)}{4z^2}\Big)\Phi_{k,1}^{(1)} = 0\ , \qquad \Big(\Box - \frac{(2k+3)(2k+5)}{4z^2}\Big)\Phi_{k,1}^{(2)} = 0\ , \tag{4.19}$$

determining the spectra of these superfields.

Note that the operator A is equal to zero (i.e. $X = \frac{1}{4}$) only for massless representations which can be realized as irreducible representations of the conformal algebra [27, 28] ($so(3,2)$ in the case of AdS_3). From the above spectra one can see that the mass terms, i.e. the eigenvalues of the operator A, are never equal to zero. That means, in particular, that the fluctuation modes for the compactification of IIB supergravity on S^3 do not satisfy the conformally invariant equations of motion in AdS space.

5 Light cone superstring action in $AdS_3 \times S^3$ R-R background

In this Section we shall find the form of the type IIB superstring action in $AdS_3 \times S^3 \times T^4$ background with R-R 3-form flux in the light-cone gauge. The Green-Schwarz action for a superstring background was constructed in [14–16] following a similar construction for $AdS_5 \times S^5$ case in [3]. Our discussion of light-cone gauge fixing will also repeat closely the same steps as in refs. [19, 20] where the $AdS_5 \times S^5$ case was treated.

In flat space superstring light-cone gauge fixing procedure in flat space consists of the two stages:

(I) fermionic light-cone gauge choice, i.e. fixing the κ-symmetry by $\Gamma^+\theta^I = 0$

(II) bosonic light-cone gauge choice, i.e. using the conformal gauge[7] $\sqrt{g}g^{\mu\nu} = \eta^{\mu\nu}$ and fixing the residual conformal diffeomorphism symmetry by $x^+(\tau,\sigma) = p^+\tau$.

Our fermionic κ-symmetry light-cone gauge will be different from the naive $\Gamma^+\theta^I = 0$ but will be related to it in the flat space limit. It will reduce the 16 fermionic coordinates θ^I_α to 8 physical Grassmann variables: "linear" θ^i and "nonlinear" η^i and their Hermitian conjugates θ_i and η_i. As in the case of the superparticle the 2-d fields θ^i, θ_i and η^i, η_i transform according to the fundamental representations of $\widetilde{SU}(2)$ and $SU(2)$ respectively. The superconformal algebra $psu(1,1|2) \oplus \widetilde{psu}(1,1|2)$ dictates that these variables should be related to the Poincaré and conformal supersymmetry in the light-cone gauge description of the boundary theory. As in the case of superparticle the superstring action and symmetry generators will have simple (quadratic) dependence on θ^i, but complicated (quartic) dependence on η^i. [8] The light-cone gauge action can be found in two related forms. One of them corresponds to the choice of the Wess-Zumino type gauge in superspace while another is based on the Killing gauge. These "gauges" or "parametrizations" do not reduce the number of fermionic degrees of freedom but only specialize a choice of fermionic coordinates.

5.1 Fermionic light-cone gauge action in WZ parametrization

Let us consider first fixing fermionic light-cone gauge in the action written in the WZ parametrization. This action turns out to be more convenient for reformulation of superstring action in terms of 2-d Dirac spinors (see next Section). Using the parametrization of the basic supercoset $[PSU(1,1|2) \times \widetilde{PSU}(1,1|2)]/[SO(2,1) \times SO(3)]$ described in Appendix E and fixing a light-cone gauge the $AdS_3 \times S^3$ superstring Lagrangian can be written as the sum of the bosonic term, term quadratic in fermions and quartic fermionic term

$$\mathcal{L} = \mathcal{L}_B + \mathcal{L}_F^{(2)} + \mathcal{L}_F^{(4)} \ . \tag{5.1}$$

Here

$$\mathcal{L}_B = -\sqrt{g}g^{\mu\nu}\Big[e^{2\phi}\partial_\mu x^+\partial_\nu x^- + \frac{1}{2}\partial_\mu\phi\partial_\nu\phi + \frac{1}{2}e_\mu^{A'}e_\nu^{A'}\Big] , \tag{5.2}$$

where $e_\mu^{A'}$ is the projection of the vielbein of S^3 which in the special parametrization we will be using is given by

$$e_\mu^{A'} = -\frac{\mathrm{i}}{2}Tr(\sigma^{A'}\partial_\mu UU^{-1}) + \frac{\mathrm{i}}{2}Tr(\sigma^{A'}\partial_\mu\tilde{U}\tilde{U}^{-1}) \ , \tag{5.3}$$

$$U^i{}_j \equiv (e^y)^i{}_j \ , \quad \tilde{U}^i{}_j \equiv (e^{-y})^i{}_j \ , \qquad U^\dagger U = I \ , \quad \tilde{U}^\dagger\tilde{U} = I \ , \tag{5.4}$$

where the trace is over $i,j = 1,2,3$. The matrices $U \in SU(2)$, $\tilde{U} \in \widetilde{SU}(2)$ depends on 3 independent coordinates $y^{A'}$

[7]We use Minkowski signature 2-d world sheet metric $g_{\mu\nu}$ with $g \equiv -\mathrm{det}g_{\mu\nu}$.

[8]Note that it is these fermionic coordinates that are most suitable for light-cone gauge fixing of kappa symmetry in AdS space, both in the superparticle and superstring cases. These coordinates were introduced in [21] in the study of light-cone gauge dynamics of superparticle in $AdS_5 \times S^5$. Light-cone gauge superstring action in $AdS_5 \times S^5$ written in terms of these coordinates was found in [19].

$$y^i{}_j \equiv \frac{\mathrm{i}}{2} y^{A'}(\sigma^{A'})^i{}_j \ , \qquad (y^i{}_j)^* = -y^j{}_i \, , \qquad y^i{}_i = 0 \, , \tag{5.5}$$

where $\sigma^{A'}$ are 3 Pauli matrices. The quadratic part of the fermionic action is

$$\mathcal{L}_F^{(2)} = e^{2\phi}\partial_\mu x^+ \Big[\frac{\mathrm{i}}{2}\sqrt{g}g^{\mu\nu}(-\theta_i\tilde{\mathcal{D}}_\nu\theta^i - \eta_i\mathcal{D}_\nu\eta^i + \mathrm{i}\eta_i e^i_{\nu j}\eta^j) + \epsilon^{\mu\nu}\eta^i C'_{ij}\tilde{\mathcal{D}}_\nu\theta^j\Big] + h.c. \ . \tag{5.6}$$

The $\epsilon^{\mu\nu}$ dependent (P-odd) term in (5.6) came from the WZ term in the covariant GS action on the supercoset. We used the following notation

$$\mathcal{D}\eta^i = d\eta^i - \Omega^i{}_j\eta^j \, , \quad \mathcal{D}\eta_i = d\eta_i + \eta_j\Omega^j{}_i \, , \quad \tilde{\mathcal{D}}\theta^i = d\theta^i - \tilde{\Omega}^i{}_j\theta^j \, , \quad \tilde{\mathcal{D}}\theta_i = d\theta_i + \theta_j\tilde{\Omega}^j{}_i \, , \tag{5.7}$$

$$e^i{}_j \equiv (\sigma^{A'})^i{}_j e^{A'} \ , \tag{5.8}$$

and $\mathcal{D} = d\sigma^\mu\mathcal{D}_\mu$, $e^i{}_j = d\sigma^\mu e^i_{\mu j}$ where $\sigma^\mu = (\tau, \sigma)$ are 2-d coordinates. $\mathcal{D}$, $\tilde{\mathcal{D}}$ are the generalized spinor derivatives on S^3. They have the structure $\mathcal{D} = d + \Omega^i{}_j J^j{}_i$, $\tilde{\mathcal{D}} = d + \tilde{\Omega}^i{}_j\tilde{J}^j{}_i$ and satisfy the relation $\mathcal{D}^2 = 0$, $\tilde{\mathcal{D}}^2 = 0$. The connections $\Omega^i{}_j$, $\tilde{\Omega}^i{}_j$ are given by

$$\Omega = dUU^{-1} \, , \quad \tilde{\Omega} = d\tilde{U}\tilde{U}^{-1} \, , \qquad d\Omega - \Omega\wedge\Omega = 0 \ , \quad d\tilde{\Omega} - \tilde{\Omega}\wedge\tilde{\Omega} = 0 \ , \tag{5.9}$$

and can be written in terms of the S^3 spin connection $\omega^{A'B'}$ and the 3-bein $e^{A'}$ as follows

$$\Omega^i{}_j = -\frac{1}{4}(\sigma^{A'B'})^i{}_j\omega^{A'B'} + \frac{\mathrm{i}}{2}(\sigma^{A'})^i{}_j e^{A'} \, , \qquad \tilde{\Omega}^i{}_j = -\frac{1}{4}(\sigma^{A'B'})^i{}_j\omega^{A'B'} - \frac{\mathrm{i}}{2}(\sigma^{A})^i{}_j e^{A'} \ . \tag{5.10}$$

C'_{ij} is the constant charge conjugation matrix of the $SO(3)$ Dirac matrix algebra (see Appendix A). The Hermitean conjugation rules are: $\theta_i^\dagger = \theta^i$, $\eta_i^\dagger = \eta^i$.

The quartic fermionic term in (5.1) depends only on half of the Grassmann variables – on η but not on θ

$$\mathcal{L}_F^{(4)} = 2\sqrt{g}g^{\mu\nu}e^{4\phi}\partial_\mu x^+\partial_\nu x^+(\eta^i\eta_i)^2 \tag{5.11}$$

5.2 "2-d spinor" form of the action

Like in the flat space case [34] and in the "long string" cases in $AdS_5 \times S^5$ [9] the resulting action can then be put into the "2-d spinor" form, where the 4+4 fermionic degrees of freedom are organized into 2 Dirac 2-d spinors, defined in *curved* 2-d geometry (we shall follow similar discussion in $AdS_5 \times S^5$ case in [19]). Such action may be useful to establishing a relation to NSR formulation.

In order to do that one needs to impose, addition to fermionic light-cone gauge, the bosonic light-cone gauge. Using the following light-cone gauge [35]

$$x^+ = \tau \ , \qquad \sqrt{g}g^{\mu\nu} = \mathrm{diag}(-e^{-2\phi}, e^{2\phi}) \ . \tag{5.12}$$

we can write the kinetic term (5.6) as

$$\mathcal{L}_F^{(2)} = \frac{\mathrm{i}}{2}(\theta_i \tilde{\mathcal{D}}_0 \theta^i + \eta_i \tilde{\mathcal{D}}_0 \eta^i - 2\mathrm{i}\eta_i e^i_{0j}\eta^j) + e^{2\phi}\eta^i C'_{ij}\tilde{\mathcal{D}}_1\theta^j + h.c. \ , \tag{5.13}$$

where we used the relation $\mathcal{D}\eta^i = \tilde{\mathcal{D}}\eta^i - \mathrm{i}e^i{}_j\eta^j$ (see (5.10)). Introducing a 2-d zweibein corresponding to the metric in (5.12)

$$e^m_\mu = \mathrm{diag}(e^{2\phi}, 1)\,, \qquad g_{\mu\nu} = -e^0_\mu e^0_\nu + e^1_\mu e^1_\nu \ , \tag{5.14}$$

we can put (5.13) in the 2-d form as follows

$$e^{-1}\mathcal{L}_F^{(2)} = -\frac{\mathrm{i}}{2}\bar\psi\varrho^m e^\mu_m \tilde{\mathcal{D}}_\mu\psi + \frac{\mathrm{i}}{2}\bar\psi\psi\partial_1\phi - \sqrt{2}\bar\psi_i e^i_{0j}\varrho^-\psi^j + h.c. \ , \tag{5.15}$$

where ϱ^m are 2-d Dirac matrices,

$$\varrho^0 = \mathrm{i}\sigma_2, \quad \varrho^1 = \sigma_1, \quad \varrho^3 = \varrho^0\varrho^1 = \sigma_3\,, \quad \varrho^\pm \equiv \frac{1}{\sqrt{2}}(\varrho^3 \pm \varrho^0)\,, \tag{5.16}$$

$\bar\psi_i = (\psi^i)^\dagger\varrho^0$, $\bar\psi\psi$ stands for $\bar\psi_i\psi^i$, ψ^T denotes the transposition of 2-d spinor and ψ's are related to the original (2-d scalar) fermionic variables θ's and η's by[9]

$$\psi^i = \begin{pmatrix} \psi^i_1 \\ \psi^i_2 \end{pmatrix}, \qquad \psi^i_1 = \frac{1}{\sqrt{2}}[\theta^i - \mathrm{i}(C'^{-1})^{ij}\eta_j]\,, \qquad \psi^i_2 = \frac{1}{\sqrt{2}}[\theta^i + \mathrm{i}(C'^{-1})^{ij}\eta_j]\,. \tag{5.17}$$

The quartic interaction term (5.11) then takes the following form

$$e^{-1}\mathcal{L}_F^{(4)} = -(\bar\psi_i\varrho^-\psi^i)^2 \ . \tag{5.18}$$

The total action is thus a kind of G/H bosonic sigma model coupled to a Thirring-type 2-d fermionic model in curved 2-d geometry (5.14) (determined by the profile of the radial function of the AdS space), and coupled to some 2-d vector fields. The interactions are such that they ensure the quantum 2-d conformal invariance of the total model [3].

The mass term $\bar\psi\psi\partial_1\phi$ in (5.15) is similar to the one in [9] (where the background string configuration was non-constant only in the radial ϕ direction) and has its origin in the $\epsilon^{\mu\nu}e^{2\phi}\partial_\mu x^+\partial_\nu\phi\eta^i C'_{ij}\theta^j$ term appearing after $\eta \leftrightarrow \theta$ symmetrization of the $\epsilon^{\mu\nu}$ term in (5.6) (its 'non-covariance' is thus a consequence of the choice $x^+ = \tau$). The action is symmetric under shifting $\psi^i \to \psi^i + \varrho^-\epsilon^i$, where ϵ^i is the 2-d Killing spinor. This symmetry reflects the fact that our original action is symmetric under shifting θ^i by a Killing spinor on S^3.

Note also that the 2-d Lorentz invariance is preserved by the fermionic light-cone gauge (original GS fermions θ are 2-d scalars) but is broken by our special choice of the bosonic gauge (5.12). The special form of $g_{\mu\nu}$ in (5.12) implies "non-covariant" dependence on ϕ in the bosonic part of the action: the action (5.2) for the field ϕ and the 3-sphere coordinates $y^{A'}$ has the form

$$\mathcal{L}_B = \frac{1}{2}e^{-2\phi}\dot\phi^2 - \frac{1}{2}e^{2\phi}\acute\phi^2 + \frac{1}{2}G_{\mathcal{AB}}(e^{-2\phi}\dot y^{\mathcal{A}}\dot y^{\mathcal{B}} - e^{2\phi}\acute y^{\mathcal{A}}\acute y^{\mathcal{B}})\,, \tag{5.19}$$

[9] In our notation $\mathrm{i}\bar\psi_i\varrho^m\nabla_m\psi^i = -\mathrm{i}\psi_1^\dagger(\nabla_0 - \nabla_1)\psi_1 - \mathrm{i}\psi_2^\dagger(\nabla_0 + \nabla_1)\psi_2, \quad \nabla_m = e^\mu_m\partial_\mu.$

where $G_{\mathcal{AB}}$ is the metric of 5-sphere ($y^{\mathcal{A}}$ are coordinates of S^3). A consequence of the unusual $g_{\mu\nu}$ gauge choice in (5.12) compared to the standard conformal gauge is that now the S^3 part of the action is no longer decoupled from the radial AdS_3 direction ϕ.

5.3 Fermionic light-cone gauge action in Killing parametrization

Now let us consider the string action in the Killing parametrization. The action is again formulated in terms of 6 bosonic coordinates $(x^{\pm}, \phi, y^{\mathcal{A}})$ ($\mathcal{A}$ label 3 independent coordinates of S^3) in terms of which the metric of $AdS_3 \times S^3$ is

$$ds^2 = 2e^{2\phi}dx^+dx^- + d\phi^2 + G_{\mathcal{AB}}(y)dy^{\mathcal{A}}dy^{\mathcal{B}} \ , \tag{5.20}$$

and 8 fermionic coordinates (θ^i, θ_i), (η^i, η_i) in the fundamental representations of $\widetilde{SU}(2)$ and $SU(2)$ respectively. In contrast to the WZ parametrization, the fermions in the Killing parametrization transform in the linear representations of $SU(2)$ and $\widetilde{SU}(2)$, and thus the covariant derivatives in WZ case (5.7) here will become ordinary derivatives. The Lagrangian is given by the sum of the "kinetic" and "Wess-Zumino" terms (see Appendices A and B for notation)

$$\mathcal{L} = \mathcal{L}_{kin} + \mathcal{L}_{WZ} \ ,$$
$$\mathcal{L}_{kin} = -\sqrt{g}g^{\mu\nu}\Big[e^{2\phi}\partial_\mu x^+\partial_\nu x^- + \frac{1}{2}\partial_\mu\phi\partial_\nu\phi + \frac{1}{2}G_{\mathcal{AB}}(y)D_\mu y^{\mathcal{A}}D_\nu y^{\mathcal{B}}\Big]$$
$$-\ \frac{\mathrm{i}}{2}\sqrt{g}g^{\mu\nu}e^{2\phi}\partial_\mu x^+\Big[\theta^i\partial_\nu\theta_i + \theta_i\partial_\nu\theta^i + \eta^i\partial_\nu\eta_i + \eta_i\partial_\nu\eta^i + \mathrm{i}e^{2\phi}\partial_\nu x^+(\eta^2)^2\Big] \ , \tag{5.21}$$

$$\mathcal{L}_{WZ} = \epsilon^{\mu\nu}e^{2\phi}\partial_\mu x^+\eta^i C^U_{ij}\partial_\nu\theta^j + h.c. \ , \tag{5.22}$$

where

$$D_\mu y^{\mathcal{A}} = \partial_\mu y^{\mathcal{A}} - 2\mathrm{i}\eta_i(V^{\mathcal{A}})^i{}_j\eta^j e^{2\phi}\partial_\mu x^+ \ , \qquad C^U_{ij} \equiv U^k{}_i C'_{kl}\tilde{U}^l{}_j \ . \tag{5.23}$$

Here $G_{\mathcal{AB}}$ and $(V^{\mathcal{A}})^i{}_j$ are the metric tensor and the Killing vectors of S^3 respectively (see Appendix A). This form of the superstring action (which we shall call "intermediate") is most convenient for deriving other forms which differ in the way one chooses the bosonic coordinates that parametrize $AdS_3 \times S^3$. For example, a useful form of the action is found by introducing a unit 4-vector u^M defined

$$u^{A'} = n^{A'}\sin|y| \ , \qquad u^4 = -\cos|y| \ , \tag{5.24}$$

in terms of which the $AdS_3 \times S^3$ metric is

$$ds^2 = e^{2\phi}dx^a dx^a + d\phi^2 + du^M du^M \ , \qquad u^M u^M = 1 \ . \tag{5.25}$$

Then the string action takes the form

$$\mathcal{L}_{kin} = -\sqrt{g}g^{\mu\nu}\Big[e^{2\phi}\partial_\mu x^+\partial_\nu x^- + \frac{1}{2}\partial_\mu\phi\partial_\nu\phi + \frac{1}{2}D_\mu u^M D_\nu u^M\Big]$$

$$-\frac{\mathrm{i}}{2}\sqrt{g}g^{\mu\nu}e^{2\phi}\partial_\mu x^+\Big[\theta^i\partial_\nu\theta_i+\theta_i\partial_\nu\theta^i+\eta^i\partial_\nu\eta_i+\eta_i\partial_\nu\eta^i+\mathrm{i}e^{2\phi}\partial_\nu x^+(\eta^2)^2\Big]\ , \qquad (5.26)$$

$$\mathcal{L}_{WZ}=\epsilon^{\mu\nu}e^{2\phi}\partial_\mu x^+(\eta^i y_{ij}\partial_\nu\theta^j+\eta_i y^{ij}\partial_\nu\theta_j)\ , \qquad (5.27)$$

where we used the relation $C^U_{ij}=-\mathrm{i}C'_{ik}u^k{}_j$, made the rescalings $\eta^i\to\mathrm{i}\eta^i$, $\eta_i\to-\mathrm{i}\eta_i$ and introduced the following notation

$$u^i{}_j\equiv(\sigma^M)^i{}_j u^M\ ,\qquad y_{ij}\equiv C'_{ik}u^k{}_j\ ,\qquad y^{ij}\equiv-u^i{}_k(C'^{-1})^{kj}\ , \qquad (5.28)$$

$$D_\mu u^M=\partial_\mu u^M-2\mathrm{i}\eta_i(R^M)^i{}_j\eta^j e^{2\phi}\partial_\mu x^+\ ,\qquad (R^M)^i{}_j=-\frac{1}{2}(\sigma^{MN})^i{}_j u^N\ . \qquad (5.29)$$

with σ^{MN} defined in (A.4). Note that $u^i{}_j$, y_{ij}, y^{ij} transform in the fundamental representation of $SU(2)$ with respect to the index i and in fundamental representation of $\widetilde{SU}(2)$ with respect to the index j. They satisfy

$$y^*_{ij}=-y^{ij}\ ,\quad u^{i*}_j=\bar{u}^j{}_i\ ,\quad \bar{u}^i{}_j\equiv(\bar{\sigma}^M)^i{}_j u^M\ ,\quad \bar{u}^i{}_k(C'^{-1})^{kj}=u^j{}_k(C'^{-1})^{ki}\ . \qquad (5.30)$$

The parametrization based on u^M is the most convenient one for the discussion of superparticle in $AdS_3\times S^3$ and of harmonic decomposition of the light-cone superfield of type IIB supergravity into the Kaluza-Klein modes (see Sections 3,4). We shall use this parametrization in the study of the light-cone superstring Hamiltonian in Section 6.

The superstring Lagrangian (5.21),(5.22) taken in any of its forms mentioned above can be represented in the following way

$$\mathcal{L}=\mathcal{L}_1+\mathcal{L}_2\ , \qquad (5.31)$$

$$\mathcal{L}_1=-h^{\mu\nu}\partial_\mu x^+\partial_\nu x^-+\partial_\mu x^+A^\mu+\frac{1}{2}h^{\mu\nu}\partial_\mu x^+\partial_\nu x^+B-\frac{1}{2}h^{\mu\nu}g_{\mathcal{AB}}D_\mu y^{\mathcal{A}}D_\nu y^{\mathcal{B}}\ , \qquad (5.32)$$

$$\mathcal{L}_2=-\frac{1}{2}h^{\mu\nu}e^{-2\phi}\partial_\mu\phi\partial_\nu\phi+T\ , \qquad (5.33)$$

where

$$g_{\mathcal{AB}}\equiv e^{-2\phi}G_{\mathcal{AB}}\ ,\qquad D_\mu y^{\mathcal{A}}\equiv\partial_\mu y^{\mathcal{A}}+F^{\mathcal{A}}\partial_\mu x^+\ , \qquad (5.34)$$

and $h^{\mu\nu}$ is defined by

$$h^{\mu\nu}\equiv\sqrt{g}g^{\mu\nu}e^{2\phi}\ ,\qquad h^{00}h^{11}-(h^{01})^2=-e^{4\phi}\ . \qquad (5.35)$$

The decomposition (5.31) is made so that the functions A^μ, B, $F^{\mathcal{A}}$ depend (i) on the anticommuting coordinates and their derivatives with respect to both τ and σ, and (ii) on the bosonic coordinates and their derivatives with respect to the world sheet spatial coordinate σ only. The reason for this decomposition is that below we shall use the phase space description with respect to the bosonic coordinates only, i.e. we shall not make the Legendre transformation with respect to the fermionic coordinates.

In the case of the "intermediate" form of the action (5.21),(5.22) these functions are

$$A^\mu=-\frac{\mathrm{i}}{2}h^{\mu\nu}(\theta^i\partial_\nu\theta_i+\eta^i\partial_\nu\eta_i)+\epsilon^{\mu1}e^{2\phi}\eta^iC^U_{ij}\acute{\theta}^j+h.c.\ , \qquad (5.36)$$

$$B=e^{2\phi}(\eta^2)^2,\qquad F^{\mathcal{A}}=-2\mathrm{i}e^{2\phi}\eta_i(V^{\mathcal{A}})^i{}_j\eta^j,\qquad T=-e^{2\phi}\acute{x}^+\eta^iC^U_{ij}\dot{\theta}^j+h.c.\,. \qquad (5.37)$$

6 Light cone Hamiltonian approach to superstring in $AdS_3 \times S^3$

Our next task is to fix the bosonic part of the light-cone gauge. We shall use the generalization of the phase space GGRT approach [36] to a curved AdS-type space described in [20], fixing the diffeomorphisms in $AdS_3 \times S^3$ cases by the *same* gauge condition as in flat space. Most of the discussion below will follow closely Ref. [20].

6.1 Phase space Lagrangian

Computing the canonical momenta for the bosonic coordinates

$$\mathcal{P}_a = \frac{\partial \mathcal{L}}{\partial \dot{x}^a} , \qquad \Pi = \frac{\partial \mathcal{L}}{\partial \dot{\phi}} , \qquad \mathcal{P}_{\mathcal{A}} = \frac{\partial \mathcal{L}}{\partial \dot{y}^{\mathcal{A}}} , \tag{6.1}$$

we get from (5.31)

$$\Pi = -h^{00} e^{-2\phi} \dot{\phi}^+ - h^{01} e^{-2\phi} \acute{\phi}^+ , \tag{6.2}$$

$$\mathcal{P}^+ = -h^{00} \dot{x}^+ - h^{01} \acute{x}^+ , \tag{6.3}$$

$$\mathcal{P}^{\mathcal{A}} = -h^{00} \dot{y}^{\mathcal{A}} - h^{01} \acute{y}^{\mathcal{A}} + F^{\mathcal{A}} \mathcal{P}^+ , \tag{6.4}$$

$$\mathcal{P}^- = -h^{00} \dot{x}^- - h^{01} \acute{x}^- + A^0 - B\mathcal{P}^+ + \mathcal{P}_{\mathcal{A}} F^{\mathcal{A}} . \tag{6.5}$$

where $\mathcal{P}^\pm \equiv \mathcal{P}_\mp$, $\mathcal{P}^{\mathcal{A}} \equiv g^{\mathcal{AB}} \mathcal{P}_{\mathcal{B}}$. By applying the same procedure as in the bosonic case we find then the following phase space Lagrangian $\mathcal{L} = \mathcal{L}_1 + \mathcal{L}_2$ (see [20])

$$\begin{aligned}
\mathcal{L}_1 &= \mathcal{P}^+ \dot{x}^- + \mathcal{P}^- \dot{x}^+ + \mathcal{P}_{\mathcal{A}} \dot{y}^{\mathcal{A}} + \frac{1}{2h^{00}} \Big[2\mathcal{P}^+ \mathcal{P}^- + 2e^{4\phi} \acute{x}^+ \acute{x}^- \\
&+ g^{\mathcal{AB}} \mathcal{P}_{\mathcal{A}} \mathcal{P}_{\mathcal{B}} + e^{4\phi} g_{\mathcal{AB}} D_1 y^{\mathcal{A}} D_1 y^{\mathcal{B}} + (\mathcal{P}^{+2} - e^{4\phi} \acute{x}^{+2}) B - 2F^{\mathcal{A}} \mathcal{P}_{\mathcal{A}} \mathcal{P}^+ \Big] \\
&+ \frac{h^{01}}{h^{00}} (\mathcal{P}^+ \acute{x}^- + \mathcal{P}^- \acute{x}^+ + \mathcal{P}_{\mathcal{A}} \acute{y}^{\mathcal{A}}) - \frac{1}{h^{00}} (\mathcal{P}^+ + h^{01} \acute{x}^+) A^0 + \acute{x}^+ A^1 ,
\end{aligned} \tag{6.6}$$

$$\mathcal{L}_2 = \Pi \dot{\phi} + \frac{1}{2h^{00}} e^{2\phi} (\Pi^2 + \acute{\phi}^2) + \frac{h^{01}}{h^{00}} \Pi \acute{\phi} + T . \tag{6.7}$$

Next, we impose the light-cone gauge

$$x^+ = \tau , \qquad \mathcal{P}^+ = p^+ . \tag{6.8}$$

Using these gauge conditions in the action and integrating over $\mathcal{P}^-$ we get the expression for h^{00}

$$h^{00} = -p^+ . \tag{6.9}$$

Inserting this into (6.6), (6.7) we get the following general form of the phase space light-cone Lagrangian[10]

$$\mathcal{L}_1 = \mathcal{P}_{\mathcal{A}} \dot{y}^{\mathcal{A}} - \frac{1}{2p^+} \Big(g^{\mathcal{AB}} \mathcal{P}_{\mathcal{A}} \mathcal{P}_{\mathcal{B}} + e^{4\phi} g_{\mathcal{AB}} \acute{y}^{\mathcal{A}} \acute{y}^{\mathcal{B}} + p^{+2} B - 2p^+ F^{\mathcal{A}} \mathcal{P}_{\mathcal{A}} \Big)$$

[10] Note that the function T in (5.37) is equal to zero in the light-cone gauge (6.8).

$$- \frac{h^{01}}{p^+}(p^+ \acute{x}^- + \mathcal{P}_{\mathcal{A}} \acute{y}^{\mathcal{A}}) + A^0 \ , \tag{6.10}$$

$$\mathcal{L}_2 = \Pi\dot{\phi} - \frac{1}{2p^+} e^{2\phi}(\Pi^2 + \acute{\phi}^2) - \frac{h^{01}}{p^+}\Pi\acute{\phi} \ . \tag{6.11}$$

This general form of the phase space Lagrangian can be specialized to different choices of bosonic coordinates by using the corresponding functions A^0, B, and $F^{\mathcal{A}}$. For the "intermediate" case (5.21),(5.22) these functions are given by (5.36),(5.37) so that we get

$$\begin{aligned} \mathcal{L} = \mathcal{L}_1 + \mathcal{L}_2 &= \Pi\dot{\phi} + \mathcal{P}_{\mathcal{A}}\dot{y}^{\mathcal{A}} + \frac{\mathrm{i}}{2}p^+(\theta^i\dot{\theta}_i + \eta^i\dot{\eta}_i + \theta_i\dot{\theta}^i + \eta_i\dot{\eta}^i) \\ &- \frac{e^{2\phi}}{2p^+}\Big[\Pi^2 + \acute{\phi}^2 + 2l^2 + G_{\mathcal{A}\mathcal{B}}\acute{y}^{\mathcal{A}}\acute{y}^{\mathcal{B}} + p^{+2}(\eta^2)^2 + 4p^+\eta_i l^i{}_j\eta^j\Big] \\ &+ e^{2\phi}(\eta^i C^U_{ij}\acute{\theta}^j + \eta_i C^{ij}_U\acute{\theta}_j) \\ &- \frac{h^{01}}{p^+}\Big[p^+\acute{x}^- + \Pi\acute{\phi} + \mathcal{P}_{\mathcal{A}}\acute{y}^{\mathcal{A}} + \frac{\mathrm{i}}{2}p^+(\theta^i\acute{\theta}_i + \eta^i\acute{\eta}_i + \theta_i\acute{\theta}^i + \eta_i\acute{\eta}^i)\Big] \ . \end{aligned} \tag{6.12}$$

Here $C^{ij}_U = -(C^U_{ij})^*$, and we used the relation

$$G^{\mathcal{A}\mathcal{B}}\mathcal{P}_{\mathcal{A}}\mathcal{P}_{\mathcal{B}} = 2l^2 \ , \qquad l^i{}_j \equiv \mathrm{i}(V^{\mathcal{A}})^i{}_j\mathcal{P}_{\mathcal{A}} \, , \qquad l^2 \equiv l^i{}_j l^j{}_i \, . \tag{6.13}$$

By applying a coordinate transformation one gets the phase space Lagrangian corresponding to the case (5.26),(5.27) in which the S^3 part is parametrized by the unit 4-vector u^M

$$\begin{aligned} \mathcal{L} &= \Pi\dot{\phi} + \mathcal{P}_M\dot{u}^M + \frac{\mathrm{i}}{2}p^+(\theta^i\dot{\theta}_i + \eta^i\dot{\eta}_i + \theta_i\dot{\theta}^i + \eta_i\dot{\eta}^i) \\ &- \frac{e^{2\phi}}{2p^+}\Big[\Pi^2 + \acute{\phi}^2 + 2l^2 + \acute{u}^M\acute{u}^M + p^{+2}(\eta^2)^2 + 4p^+\eta_i l^i{}_j\eta^j\Big] \\ &+ e^{2\phi}(\eta^i y_{ij}\acute{\theta}^j + \eta_i y^{ij}\acute{\theta}_j) \\ &- \frac{h^{01}}{p^+}\Big[p^+\acute{x}^- + \Pi\acute{\phi} + \mathcal{P}_M\acute{u}^M + \frac{\mathrm{i}}{2}p^+(\theta^i\acute{\theta}_i + \eta^i\acute{\eta}_i + \theta_i\acute{\theta}^i + \eta_i\acute{\eta}^i)\Big] \ , \end{aligned} \tag{6.14}$$

where $\mathcal{P}_M$ is the canonical momentum for u^M and $l^i{}_j$ in (6.13) has the following explicit form

$$l^i{}_j = \frac{\mathrm{i}}{2}(\sigma^{MN})^i{}_j u^M\mathcal{P}^N \ . \tag{6.15}$$

Here and below $l^i{}_j$ is for the classical orbital momentum (note that going to the superparticle limit, after the quantization we get $\mathcal{P}^M = -\mathrm{i}\hat{\partial}^M$ and then the classical orbital momentum $l^i{}_j$ (6.15) becomes the quantum momentum $l^i{}_j$ in (3.9)). Taking into account the constraint $u^M\mathcal{P}^M = 0$ (see (6.32)) we get

$$l^i{}_k l^k{}_j = \frac{1}{4}\mathcal{P}^M\mathcal{P}^M\delta^i{}_j \, , \qquad l^2 = \frac{1}{2}\mathcal{P}^M\mathcal{P}^M \ . \tag{6.16}$$

The above Lagrangian leads to the following (minus) Hamiltonian

$$P^- = \int_0^1 d\sigma \ \mathcal{P}^- \ , \tag{6.17}$$

where the Hamiltonian density $-\mathcal{P}^-$ is given by

$$\mathcal{P}^- = -\frac{e^{2\phi}}{2p^+}\Big[\Pi^2 + \acute{\phi}^2 + 2l^2 + \acute{u}^M\acute{u}^M + p^{+2}(\eta^2)^2 + 4p^+\eta_i l^i{}_j\eta^j\Big] + e^{2\phi}(\eta^i y_{ij}\acute{\theta}^j + \eta_i y^{ij}\acute{\theta}_j)\,. \tag{6.18}$$

It should be supplemented by the constraint

$$p^+\acute{x}^- + \Pi\acute{\phi} + \mathcal{P}_M\acute{u}^M + \frac{\mathrm{i}}{2}p^+(\theta^i\acute{\theta}_i + \eta^i\acute{\eta}_i + \theta_i\acute{\theta}^i + \eta_i\acute{\eta}^i) = 0\,. \tag{6.19}$$

As usual, this constraint allows one to express the non-zero modes of the bosonic coordinate x^- in terms of the transverse physical ones.

It is easy to see that in the particle theory limit the superstring Hamiltonian (6.18) reduces to the superparticle one which was found in section 3 by applying the direct method of constructing relativistic dynamics [24] based on the symmetry algebra. Indeed, the (quantum, operator-ordered) superparticle light-cone Hamiltonian in (3.14),(3.19) can be rewritten as follows

$$\mathcal{P}^- = -\frac{1}{2p^+}\Big[e^\phi\Pi e^\phi\Pi + e^{2\phi}(2l^2 + (p^+\eta^2 - 1)^2 + 4p^+\eta_i l^i{}_j\eta^j)\Big]\,. \tag{6.20}$$

The string Hamiltonian (6.18) reduces to (6.20) modulo terms "quantum" terms proportional to η^2 and a constant (in string Hamiltonian we ignore operator ordering). The derivation of the light-cone string action from the covariant one given above thus provides, in the particle limit, also a self-contained Lagrangian derivation of the light-cone gauge superparticle Hamiltonian (3.14) (obtained indirectly from the symmetry algebra in Section 3) from a covariant action. This represents a consistency check on the two different methods used in Section 3 and in the present Section.

6.2 Equations of motion

The equations of motion corresponding to the phase space superstring Lagrangian (6.14) are

$$\dot{\phi} = \frac{e^{2\phi}}{p^+}\Pi\,, \qquad \dot{\Pi} = \frac{1}{p^+}\partial_\sigma(e^{2\phi}\acute{\phi}) + 2\mathcal{P}^-\,, \tag{6.21}$$

$$\dot{u}^M = \frac{e^{2\phi}}{p^+}\mathcal{P}^M - \mathrm{i}e^{2\phi}\eta_i(\sigma^{MN})^i{}_j\eta^j u^N\ , \tag{6.22}$$

$$\dot{\mathcal{P}}^M = -\frac{e^{2\phi}}{p^+}u^M\mathcal{P}^N\mathcal{P}^N + \frac{1}{p^+}v^{MN}\partial_\sigma(e^{2\phi}\acute{u}^N) - \mathrm{i}e^{2\phi}\eta_i(\sigma^{MN})^i{}_j\eta^j\mathcal{P}^N \tag{6.23}$$

$$+\ e^{2\phi}v^{MN}\eta^i\rho^N_{ij}\acute{\theta}^j + e^{2\phi}v^{MN}\eta_i(\rho^N)^{ij}\acute{\theta}_j \tag{6.24}$$

$$\dot{\theta}^i = \frac{\mathrm{i}}{p^+}\partial_\sigma(e^{2\phi}\eta_j y^{ji})\ , \qquad \dot{\theta}_i = \frac{\mathrm{i}}{p^+}\partial_\sigma(e^{2\phi}\eta^j y_{ji})\ , \tag{6.25}$$

$$\dot{\eta}^i = e^{2\phi}\Big[\mathrm{i}\eta^2\eta^i - \frac{2\mathrm{i}}{p^+}(l\eta)^i + \frac{\mathrm{i}}{p^+}y^{ij}\acute{\theta}_j\Big]\ , \quad \dot{\eta}_i = e^{2\phi}\Big[-\mathrm{i}\eta^2\eta_i + \frac{2\mathrm{i}}{p^+}(\eta l)_i + \frac{\mathrm{i}}{p^+}y_{ij}\acute{\theta}^j\Big], \tag{6.26}$$

where v^{MN} is given by (3.11) and, as previously, do not distinguish between the upper and lower "S^3" indices M, N, i.e. use the convention $\mathcal{P}_M = \mathcal{P}^M$. These equations can be written in the Hamiltonian form. Introducing the notation $\mathcal{X}$ for the phase space variables $(\Pi, \phi, \mathcal{P}^M, u^M, \theta^i, \theta_i, \eta^i, \eta_i)$, the Hamiltonian equations are

$$\dot{\mathcal{X}} = [\mathcal{X}, \mathcal{P}^-] \,, \tag{6.27}$$

where the phase space variables satisfy the (classical) Poisson-Dirac brackets

$$[\ \Pi(\sigma), \phi(\sigma')\ ] = \delta(\sigma, \sigma') \,, \tag{6.28}$$

$$[\mathcal{P}^M(\sigma), u^N(\sigma')] = v^{MN}\delta(\sigma,\sigma') \,, \qquad [\mathcal{P}^M(\sigma), \mathcal{P}^N(\sigma')] = (u^M\mathcal{P}^N - u^N\mathcal{P}^M)\delta(\sigma,\sigma') \,, \tag{6.29}$$

$$\{\theta_i(\sigma), \theta^j(\sigma')\} = \frac{\mathrm{i}}{p^+}\delta_i^j\delta(\sigma,\sigma') \ , \qquad \{\eta_i(\sigma), \eta^j(\sigma')\} = \frac{\mathrm{i}}{p^+}\delta_i^j\delta(\sigma,\sigma') \ , \tag{6.30}$$

$$[x_0^-, \theta^i] = \frac{1}{2p^+}\theta^i, \quad [x_0^-, \theta_i] = \frac{1}{2p^+}\theta_i \,, \quad [x_0^-, \eta^i] = \frac{1}{2p^+}\eta^i, \quad [x_0^-, \eta_i] = \frac{1}{2p^+}\eta_i \,. \tag{6.31}$$

x_0^- is the zero mode of x^- so that $[p^+, x_0^-] = 1$. All the remaining brackets are equal to zero. The structure of (6.29) reflects the fact that in the Hamiltonian formulation the condition $u^M u^M = 1$ should be supplemented by the constraint

$$u^M\mathcal{P}^M = 0 \,. \tag{6.32}$$

These are second class constraints, and the Dirac procedure leads then to the classical Poisson-Dirac brackets (6.29). To derive (6.30),(6.31) one is to take into account that the Lagrangian (6.14) has the following second class constraints

$$p_{\theta^i} + \frac{\mathrm{i}}{2}p^+\theta_i = 0 \,, \qquad p_{\theta_i} + \frac{\mathrm{i}}{2}p^+\theta^i = 0 \ , \tag{6.33}$$

where p_{θ^i}, p_{θ_i} are the canonical momenta of fermionic coordinates. The same constraints are found for the fermionic coordinates η^i, η_i. Starting with the Poisson brackets

$$\{p_{\theta^i}, \theta^j\}_{P.B.} = \delta_i^j \,, \qquad \{p_{\theta_i}, \theta_j\}_{P.B.} = \delta_j^i \,, \qquad [p^+, x_0^-]_{P.B.} = 1 \,, \tag{6.34}$$

one gets then the Poisson-Dirac brackets given in (6.30),(6.31).

7 Noether charges as generators of the superalgebra $psu(1,1|2) \oplus \widetilde{psu}(1,1|2)$

The Noether charges play an important role in the analysis of the symmetries of dynamical systems. The choice of the light-cone gauge spoils manifest global symmetries, and in order to demonstrate that these global invariances are still present one needs to find the Noether charges which generate them.[11] These charges determine the structure of

[11] In what follows "currents" and "charges" will mean both bosonic and fermionic ones, i.e. will include supercurrents and supercharges.

superstring field theory in the light-cone gauge [37]. The first step in the construction of superstring field theory is to find a free (quadratic) superfield representation of the generators of the $psu(1,1|2) \oplus \widetilde{psu}(1,1|2)$ superalgebra. The charges we obtain below can be used to obtain (after quantization) these free superstring field charges.

The Noether charges for a superparticle in $AdS_3 \times S^3$ were found in Section 3. These charges are helpful in establishing a correspondence between the bulk fields of type IIB supergravity and the chiral primary operators of the boundary theory in a manifestly supersymmetric way. Superstring Noether charges should thus be important for the study of the AdS/CFT correspondence at the full string-theory level. Our discussion below will be an adaptation to the $AdS_3 \times S^3$ case of the results for the currents in the $AdS_5 \times S^5$ case given in [20].

7.1 Currents for κ-symmetry light-cone gauge fixed superstring action

As usual, symmetry generating charges can be obtained from conserved currents. Since currents themselves may be helpful in some applications, we shall first derive them starting with the κ-symmetry gauge fixed Lagrangian in the form given in (5.26),(5.27) and using the standard Noether method based on the localization of the parameters of the associated global transformations. Let ϵ be a parameter of some global transformation which leaves the action invariant. Replacing it by a function of worldsheet coordinates τ, σ, the variation of the action takes the form

$$\delta S = \int d^2\sigma \; \mathcal{G}^\mu \partial_\mu \epsilon \,, \tag{7.1}$$

where $\mathcal{G}^\mu$ is the corresponding current. Making use of this formula, we shall find below those currents which are related to symmetries that do not involve compensating κ-symmetry transformation. The remaining currents will be found in the next subsection starting from the action (6.14) where both the κ-symmetry and the bosonic light-cone gauges are fixed.

Let us start with the translation invariance $\delta x^a = \epsilon^a$. Applying (7.1) to the Lagrangian (5.26),(5.27) gives the translation currents

$$\begin{aligned}
\mathcal{P}^{+\mu} &= -\sqrt{g} g^{\mu\nu} e^{2\phi} \partial_\nu x^+ \,, && (7.2)\\
\mathcal{P}^{-\mu} &= -\sqrt{g} g^{\mu\nu} \Big(e^{2\phi} \partial_\nu x^- + F^M D_\nu u^M \Big) \\
&- \frac{\mathrm{i}}{2} \sqrt{g} g^{\mu\nu} e^{2\phi} \Big(\theta^i \partial_\nu \theta_i + \theta_i \partial_\nu \theta^i + \eta^i \partial_\nu \eta_i + \eta^i \partial_\nu \eta^i + 2\mathrm{i} e^{2\phi} \partial_\nu x^+ (\eta^2)^2 \Big) \\
&+ \epsilon^{\mu\nu} e^{2\phi} (\eta^i y_{ij} \partial_\nu \theta^j + \eta_i y^{ij} \partial_\nu \theta_j) \,, \quad F^M \equiv \mathrm{i} \eta_i (\sigma^{MN})^i{}_j \eta^j e^{2\phi} u^N \,. && (7.3)
\end{aligned}$$

Some of the remaining bosonic currents can be expressed in terms of supercurrents. The invariance with respect to the super-transformations

$$\delta\theta^i = \epsilon^i \,, \qquad \delta\theta_i = \epsilon_i \,, \qquad \delta x^- = -\frac{\mathrm{i}}{2} \epsilon^i \theta_i - \frac{\mathrm{i}}{2} \epsilon_i \theta^i \,, \tag{7.4}$$

gives the supercurrents

$$\mathcal{Q}^{+i\mu} = -\sqrt{g}g^{\mu\nu}e^{2\phi}\theta^i\partial_\nu x^+ + \mathrm{i}\epsilon^{\mu\nu}e^{2\phi}\eta_j y^{ji}\partial_\nu x^+ \ , \tag{7.5}$$

$$\mathcal{Q}_i^{+\mu} = -\sqrt{g}g^{\mu\nu}e^{2\phi}\theta_i\partial_\nu x^+ + \mathrm{i}\epsilon^{\mu\nu}e^{2\phi}\eta^j y_{ji}\partial_\nu x^+ \ . \tag{7.6}$$

The invariance of the action (5.26),(5.27) with respect to the rotation of (super) coordinates in the (x^+, x^-) plane

$$\delta x^{\pm} = e^{\pm\epsilon}x^{\pm} \ , \qquad \delta(\theta^i, \theta_i \, , \eta^i, \eta_i) = e^{-\epsilon/2}(\theta^i, \theta_i \, , \eta^i, \eta_i) \ , \tag{7.7}$$

leads to

$$\mathcal{J}^{+-\mu} = x^+\mathcal{P}^{-\mu} - x^-\mathcal{P}^{+\mu} + \frac{\mathrm{i}}{2}\theta^i\mathcal{Q}_i^{+\mu} + \frac{\mathrm{i}}{2}\theta_i\mathcal{Q}^{+i\mu} \ , \tag{7.8}$$

while the invariance with respect to the dilatations

$$\delta x^a = e^{\epsilon}x^a \ , \qquad \delta\phi = -\epsilon \ , \qquad \delta(\theta^i, \theta_i \, , \eta^i, \eta_i) = e^{\epsilon/2}(\theta^i, \theta_i \, , \eta^i, \eta_i) \ , \tag{7.9}$$

implies conservation of the dilatation current

$$\mathcal{D}^{\mu} = x^a\mathcal{P}^{a\mu} + \sqrt{g}g^{\mu\nu}\partial_\nu\phi - \frac{\mathrm{i}}{2}\theta^i\mathcal{Q}_i^{+\mu} - \frac{\mathrm{i}}{2}\theta_i\mathcal{Q}^{+i\mu} \ . \tag{7.10}$$

The invariances with respect to the $SU(2)$ ($\epsilon^i{}_i = 0$) and $\widetilde{SU}(2)$ rotations ($\tilde{\epsilon}^i{}_i = 0$)

$$\delta y^{ij} = \epsilon^i{}_l y^{lj} \ , \quad \text{i.e.} \quad \delta u^M = -\frac{1}{2}\epsilon^i{}_j(\sigma^{MN})^j{}_i u^N, \qquad \delta\eta^i = \epsilon^i{}_j\eta^j \ , \quad \delta\eta_i = -\eta_j\epsilon^j{}_i \ , \tag{7.11}$$

$$\delta y^{ij} = \tilde{\epsilon}^j{}_l y^{il}, \quad \text{i.e.} \quad \delta u^M = -\frac{1}{2}\tilde{\epsilon}^i{}_j(\bar{\sigma}^{MN})^j{}_i u^N, \qquad \delta\theta^i = \tilde{\epsilon}^i{}_j\theta^j \ , \qquad \delta\theta_i = -\theta_j\tilde{\epsilon}^j{}_i \ , \tag{7.12}$$

give the following $SU(2)$ and $\widetilde{SU}(2)$ currents, respectively,

$$\mathcal{J}^i{}_j{}^{\mu} = -\mathrm{i}\sqrt{g}g^{\mu\nu}\frac{1}{2}(\sigma^{MN})^i{}_j u^M D_\nu u^N + (\eta^i\eta_j - \frac{1}{2}\delta^i_j\eta^2)\mathcal{P}^{+\mu} \ . \tag{7.13}$$

$$\begin{aligned}\widetilde{\mathcal{J}}^i{}_j{}^{\mu} = & - \mathrm{i}\sqrt{g}g^{\mu\nu}\frac{1}{2}(\bar{\sigma}^{MN})^i{}_j u^M D_\nu u^N + (\theta^i\theta_j - \frac{1}{2}\delta^i_j\theta^2)\mathcal{P}^{+\mu} \\ & - \mathrm{i}\epsilon^{\mu\nu}e^{2\phi}\partial_\nu x^+(\eta^l y_{lj}\theta^i - \frac{1}{2}\delta^i_j\eta^k y_{kl}\theta^l) + \mathrm{i}\epsilon^{\mu\nu}e^{2\phi}\partial_\nu x^+(\eta_l y^{li}\theta_j - \frac{1}{2}\delta^i_j\eta_k y^{kl}\theta_l) \, .\end{aligned} \tag{7.14}$$

7.2 Charges for bosonic and κ-symmetry light-cone gauge fixed superstring action

In the previous Section we have listed the (super)currents starting with the κ-symmetry light-cone gauge fixed action given in (5.26),(5.27). They can be used to find currents for the action where both the fermionic κ-symmetry and the bosonic reparametrization symmetry are fixed by the light-cone type gauges (6.14). To find the components of currents ($\mathcal{G}^0$) in the world-sheet time direction one needs to use the relations (6.2)–(6.5)

for the canonical momenta and to insert the light-cone gauge conditions (6.8) and (6.9) into the expressions for the currents. The charges are then $G = \int d\sigma\, \mathcal{G}^0$.

Let us start with the kinematical generators (charges) (3.1). The results for the currents imply the following representations

$$P^+ = p^+ , \qquad Q^{+i} = \int p^+\theta^i , \qquad Q^+_i = \int p^+\theta_i \ . \tag{7.15}$$

Note that these charges depend only on the zero modes of string coordinates (the integrands are $\mathcal{G}^0$ parts of the corresponding currents in world-sheet time direction: $\mathcal{Q}^{+i0}$, $\mathcal{Q}^{+0}_i$ and $\mathcal{P}^{+0} = p^+$). The remaining kinematical charges depend on non-zero string modes

$$J^{+-} = \int x^+\mathcal{P}^- - x^-p^+ \ , \qquad D = \int x^+\mathcal{P}^- + x^-p^+ - \Pi \, , \tag{7.16}$$

$$J^i{}_j = \int l^i{}_j + p^+\eta^i\eta_j - \frac{1}{2}\delta^i_j p^+\eta^2 \ , \qquad \widetilde{J}^i{}_j = \int \tilde{l}^i{}_j + p^+\theta^i\theta_j - \frac{1}{2}\delta^i_j p^+\theta^2 \ , \tag{7.17}$$

where $l^i{}_j$ is given by (6.15) and $\widetilde{l}^i{}_j = \frac{\mathrm{i}}{2}(\bar\sigma^{MN})^i{}_j u^M\mathcal{P}^N$. The derivation of the remaining charges follows the procedure described in Appendix D of [20]. The conformal (super)charges are given by (3.27),(3.28) where

$$S^+_i|_{x^+=0} = \int \frac{1}{\sqrt{2}} e^{-\phi}p^+\eta_i \, , \qquad S^{+i}|_{x^+=0} = \int \frac{1}{\sqrt{2}} e^{-\phi}p^+\eta^i \, , \tag{7.18}$$

$$K^+|_{x^+=0} = \int -\frac{1}{2}e^{-2\phi}p^+ \, . \tag{7.19}$$

The dynamical Poincaré charges Q^{-i}, Q^-_i and the conformal charges S^{-i}, S^-_i are

$$Q^-_i = \int \frac{e^\phi}{\sqrt{2}}\Big(\mathrm{i}\eta_i\Pi - p^+\eta^2\eta_i + 2\eta_j l^j{}_i + y_{ij}\acute{\theta}^j\Big) \ , \tag{7.20}$$

$$Q^{-i} = \int \frac{e^\phi}{\sqrt{2}}\Big(-\mathrm{i}\eta^i\Pi - p^+\eta^2\eta^i + 2l^i{}_j\eta^j - y^{ij}\acute{\theta}_j\Big) \, . \tag{7.21}$$

$$S^{-i} = \int \theta^i S - \tilde{l}^i{}_j\theta^j + \frac{1}{2}e^{-\phi}\partial_\sigma(e^\phi\eta_j)\ y^{ji} \tag{7.22}$$

$$S^-_i = \int \theta_i\bar{S} - \theta_j\tilde{l}^j{}_i - \frac{1}{2}e^{-\phi}\partial_\sigma(e^\phi\eta^j)\ y_{ji} \tag{7.23}$$

$$S = \mathrm{i}x^-p^+ - \frac{\mathrm{i}}{2}\Pi + \frac{1}{2}p^+\theta^2 \ , \qquad \bar{S} = -\mathrm{i}x^-p^+ + \frac{\mathrm{i}}{2}\Pi + \frac{1}{2}p^+\theta^2 \ . \tag{7.24}$$

Note that the $G|_{x^+=0}$ parts of the kinematical charges (3.1) can be obtained from the superparticle ones simply by replacing the particle coordinates by the string ones. The remaining dynamical generator K^- can be found by using the expressions found above and applying the commutation relations of $psu(1,1|2) \oplus \widetilde{psu}(1,1|2)$ superalgebra.

Our classical charges are normalized so that after the quantization, i.e. the replacement of the classical Poisson-Dirac brackets (6.28)-(6.31) by quantum (anti)commutators

$$[\ ,\ ]_{P.B} \to \mathrm{i}[\ ,\ ] \, , \qquad \{\ ,\ \}_{P.B} \to \mathrm{i}\{\ ,\ \} \, , \tag{7.25}$$

redefinitions $J^{+-} \to -\mathrm{i}J^{+-}$, $D \to -\mathrm{i}D$, $K^\pm \to -K^\pm$, and appropriate operator ordering the charges satisfy the commutation relations of $psu(1,1|2) \oplus \widetilde{psu}(1,1|2)$ superalgebra given in (2.8)-(2.15).

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Appendix A Notation and basic definitions

In the main part of the paper we use the following conventions for the indices:

$a, b = 0, 1$	boundary Minkowski space indices
$\mathcal{A}, \mathcal{B}, \mathcal{C} = 1, 2, 3$	S^3 coordinate space indices
$\mathrm{A}', \mathrm{B}', \mathrm{C}' = 1, 2, 3$	$so(3)$ vector indices (S^3 tangent space indices)
$M, N, K, L = 1, \ldots, 4$	$so(4)$ vector indices
$i, j, k, n = 1, 2$	$su(2)$ and $\widetilde{su}(2)$ vector indices
$\mu, \nu = 0, 1$	world-sheet coordinate indices

We decompose x^a into the light-cone coordinates $x^a = (x^+, x^-)$ where $x^\pm \equiv \frac{1}{\sqrt{2}}(x^1 \pm x^0)$. We suppress the flat space metric tensor $\eta_{ab} = (-, +)$ in scalar products, i.e. $A^a B^a \equiv \eta_{ab} A^a B^b$. The $SO(1,1)$ vector A^a is decomposed as $A^a = (A^+, A^-)$ so that the scalar product is $A^a B^a = A^+ B^- + A^- B^+$. The derivatives with respect to the world-sheet coordinates (τ, σ) are

$$\dot{x} \equiv \partial_\tau x\,, \qquad \acute{x} \equiv \partial_\sigma x \tag{A.1}$$

The world-sheet Levi-Civita $\epsilon^{\mu\nu}$ is defined with $\epsilon^{01} = 1$.

The four matrices $(\sigma^M)^i{}_j$, $(\bar{\sigma}^M)^i{}_j$ are off-diagonal blocks of the $SO(4)$ Dirac matrices γ^M in the chiral (Weyl) representation, i.e.

$$\gamma^M = \left(\begin{array}{cc} 0 & \sigma^M \\ \bar{\sigma}^M & 0 \end{array}\right), \qquad (\sigma^M)^i{}_k (\bar{\sigma}^N)^k{}_j + (\sigma^N)^i{}_k (\bar{\sigma}^M)^k{}_j = 2\delta^{MN} \delta^i_j\,, \tag{A.2}$$

$$C'_{ik} (\sigma^M)^k{}_j = C'_{jk} (\bar{\sigma}^M)^k{}_i\,, \quad (\sigma^M)^{i*}_{\ j} \equiv (\bar{\sigma}^M)^j{}_i\,, \tag{A.3}$$

where C' is a charge conjugation matrix. σ^{MN}, $\bar{\sigma}^{MN}$ are defined by

$$(\sigma^{MN})^i{}_j \equiv \frac{1}{2} (\sigma^M)^i{}_k (\bar{\sigma}^N)^k{}_j - (M \leftrightarrow N)\,, \quad (\bar{\sigma}^{MN})^i{}_j \equiv \frac{1}{2} (\bar{\sigma}^M)^i{}_k (\sigma^N)^k{}_j - (M \leftrightarrow N)\,. \tag{A.4}$$

We use the following explicit form of σ^M, $\bar{\sigma}^M$ and C'_{ij}

$$\sigma^M = (\sigma^{A'}, -\mathrm{i}I)\,, \qquad \bar{\sigma}^M = (\sigma^{A'}, \mathrm{i}I)\,, \qquad C'_{ij} = c\epsilon_{ij}\,, \quad |c| = 1\,. \tag{A.5}$$

We also use the matrices ρ^M defined by

$$\rho^M_{ij} \equiv C'_{ik} (\sigma^M)^k{}_j\,, \qquad (\rho^M)^{ij} \equiv -(\sigma^M)^i{}_k (C'^{-1})^{kj}\,, \quad (\rho^M_{ij})^* = -(\rho^M)^{ij} \tag{A.6}$$

We assume the following Hermitian conjugation rule for the fermionic coordinates and the notation for their squares

$$\theta_i^\dagger = \theta^i\,, \qquad \eta_i^\dagger = \eta^i\,, \qquad \theta^2 \equiv \theta^i\theta_i\,, \qquad \eta^2 \equiv \eta^i\eta_i\,. \tag{A.7}$$

The generators of the $su(2)$ and $\widetilde{su}(2)$ subalgebras of $so(4)$

$$[J^{MN}, J^{KL}] = \delta^{NK}J^{ML} + \text{ 3 terms} \tag{A.8}$$

are defined by

$$J^i{}_j = \frac{1}{4}(\sigma^{MN})^i{}_j J^{MN}\,, \qquad \widetilde{J}^i{}_j = \frac{1}{4}(\bar\sigma^{MN})^i{}_j J^{MN}\,. \tag{A.9}$$

The translation operator $J^{4A'}$ on S^3 is

$$J^{4A'} = \frac{\mathrm{i}}{2}(\sigma^{A'})^j{}_i(J^i{}_j - \tilde{J}^i{}_j),. \tag{A.10}$$

The coset representative of S^3 defined by $g_y \equiv \exp(y^{A'}J^{4A'})$ takes then the form given in (E.6) below. In terms of these coordinates, the 3-sphere interval, metric tensor and vielbein are given by

$$ds^2_{S^3} = d|y|^2 + \sin^2|y| ds^2_{S^2}\ , \qquad ds^2_{S^2} = dn^{\mathcal{A}}dn^{\mathcal{A}}\ , \quad n^{\mathcal{A}}n^{\mathcal{A}} = 1\ , \tag{A.11}$$

$$G_{\mathcal{AB}} = e^{A'}_{\mathcal{A}}e^{A'}_{\mathcal{B}}\ , \qquad e^{A'}_{\mathcal{A}} = \frac{\sin|y|}{|y|}(\delta^{A'}_{\mathcal{A}} - n_{\mathcal{A}}n^{A'}) + n_{\mathcal{A}}n^{A'}\ , \tag{A.12}$$

$$G_{\mathcal{AB}} = \frac{\sin^2|y|}{|y|^2}(\delta_{\mathcal{AB}} - n_{\mathcal{A}}n_{\mathcal{B}}) + n_{\mathcal{A}}n_{\mathcal{B}}\ , \qquad n^{\mathcal{A}} \equiv \frac{y^{\mathcal{A}}}{|y|}, \qquad |y| = \sqrt{y^{A'}y^{A'}}\,. \tag{A.13}$$

We use the convention $y^{\mathcal{A}} = \delta^{\mathcal{A}}_{A'}y^{A'}$ and the same for $n^{A'}$. The S^3 Killing vectors $V^{A'}$ and $V^{A'\mathrm{B}'}$ corresponding to the 3 translations and 3 $SO(3)$ rotations are

$$V^{A'} = \Big[|y|\cot|y|(\delta^{A'\mathcal{A}} - n^{A'}n^{\mathcal{A}}) + n^{A'}n^{\mathcal{A}}\Big]\partial_{y^{\mathcal{A}}}\ , \quad V^{A'\mathrm{B}'} = y^{A'}\partial_{y^{\mathrm{B}'}} - y^{\mathrm{B}'}\partial_{y^{A'}}\ . \tag{A.14}$$

They can be collected into the $SU(2)$ combination

$$(V^{\mathcal{A}})^i{}_j\partial_{y^{\mathcal{A}}} = \frac{1}{4}(\sigma^{A'\mathrm{B}'})^i{}_jV^{A'\mathrm{B}'} + \frac{\mathrm{i}}{2}(\sigma^{A'})^i{}_jV^{A'}\ . \tag{A.15}$$

These relations and (5.24) imply the following relations

$$G_{\mathcal{AB}}(\eta V^{\mathcal{A}}\eta)(\eta V^{\mathcal{B}}\eta) = (\eta R^M\eta)^2\ , \qquad G_{\mathcal{AB}}(\eta V^{\mathcal{A}}\eta)dy^{\mathcal{B}} = (\eta R^M\eta)du^M\ , \tag{A.16}$$

where R^M is defined by (5.29), which have been used to transform (5.21) to (5.26).

Appendix B Light-cone basis of $psu(1,1|2)\oplus\widetilde{psu}(1,1|2)$

Here we explain the relation between the $su(1,1)\oplus su(2)$ covariant and light-cone bases of the $psu(1,1|2)$ algebra and define the light-cone basis of $psu(1,1|2)\oplus\widetilde{psu}(1,1|2)$. We find it convenient to introduce intermediate basis defined by

$$m^{+-}\equiv -m^1{}_1\,,\qquad m^{+1}\equiv\frac{1}{\sqrt{2}}m^2{}_1\,,\qquad m^{-1}\equiv-\frac{1}{\sqrt{2}}m^1{}_2\,,\tag{B.1}$$

$$q^{+i}\equiv -q_1^i\,,\qquad q^{-i}\equiv q_2^i\,,\qquad q_i^+\equiv q_i^2\,,\qquad q_i^-\equiv q_i^1\,.\tag{B.2}$$

In this basis the Hermitean rules for supercharges take conventional form $(q^{+i})^\dagger=q_i^+$, $(q^{-i})^\dagger=q_i^-$, and for the (anti)commutators one has

$$[m^{+-},m^{\pm1}]=\pm m^{\pm1}\,,\qquad [m^{+1},m^{-1}]=-m^{+-}\,,\qquad [m^{+-},q_i^\pm]=\pm\frac{1}{2}q_i^\pm\,,\tag{B.3}$$

$$\{q^{\pm i},q_j^\pm\}=-a\sqrt{2}\delta_j^i m^{\pm1}\,,\qquad \{q^{\pm i},q_j^\mp\}=a(\delta_j^i m^{+-}\mp m^i{}_j)\,,\tag{B.4}$$

$$[q^{-i},m^{+1}]=-\frac{1}{\sqrt{2}}q^{+i}\,,\qquad [q^{+i},m^{-1}]=\frac{1}{\sqrt{2}}q^{-i}\,.\tag{B.5}$$

The light-cone basis of $psu(1,1|2)\oplus\widetilde{psu}(1,1|2)$ superalgebra is defined by

$$P^+=\sqrt{2}\tilde m^{+1}\,,\qquad P^-=\sqrt{2}m^{-1}\,,\qquad K^+=\sqrt{2}m^{+1}\,,\qquad K^-=\sqrt{2}\tilde m^{-1}\,,\tag{B.6}$$

$$J^{+-}=m^{+-}+\tilde m^{+-}\,,\qquad D=m^{+-}-\tilde m^{+-}\,,\qquad J^i{}_j=m^i{}_j\,,\qquad \tilde J^i{}_j=\tilde m^i{}_j\,,\tag{B.7}$$

$$Q^{+i}=\tilde q^{+i}\,,\qquad Q_i^+=\tilde q_i^+\,,\qquad Q^{-i}=q^{-i}\,,\qquad Q_i^-=q_i^-\,,\tag{B.8}$$

$$S^{-i}=\tilde a\tilde q^{-i}\,,\qquad S_i^-=\tilde a^*\tilde q_i^-\,,\qquad S^{+i}=aq^{+i}\,,\qquad S_i^+=a^*q_i^+\,.\tag{B.9}$$

The constants $a,\tilde a$ are chosen to be $a=-\mathrm{i}$, $\tilde a=\mathrm{i}$. Then the commutation relations are

$$\begin{gathered}
[P^\pm,K^\mp]=D\mp J^{+-}\,,\\
[D,P^\pm]=-P^\pm\,,\quad [D,K^\pm]=K^\pm\,,\quad [J^{+-},P^\pm]=\pm P^\pm\,,\quad [J^{+-},K^\pm]=\pm K^\pm\,,\\
[D,Q_i^\pm]=-\tfrac{1}{2}Q_i^\pm\,,\quad [D,S_i^\pm]=\tfrac{1}{2}S_i^\pm\,,\quad [J^{+-},Q_i^\pm]=\pm\tfrac{1}{2}Q_i^\pm\,,\quad [J^{+-},S_i^\pm]=\pm\tfrac{1}{2}S_i^\pm\,,\\
[S_i^\pm,P^\mp]=\mathrm{i}Q_i^\mp\,,\quad [Q^{\pm i},K^\mp]=-\mathrm{i}S^{\mp i},\quad \{Q^{\pm i},Q_j^\pm\}=\mp\mathrm{i}P^\pm\delta_j^i,\quad \{S^{\pm i},S_j^\pm\}=\pm\mathrm{i}K^\pm\delta_j^i\\
\{Q^{+i},S_j^-\}=\tfrac{1}{2}(J^{+-}-D)\delta_j^i-\tilde J^i{}_j\,,\qquad \{Q^{-i},S_j^+\}=\tfrac{1}{2}(J^{+-}+D)\delta_j^i+J^i{}_j\,.
\end{gathered}\tag{B.10}$$

The supercharges Q_i^-, Q^{-i}, S^{+i}, S_i^+ transform in the fundamental representation of $su(2)$ i.e. they are rotated only by $J^i{}_j$ and satisfy (2.13). The remaining supercharges Q^{+i}, Q_i^+, S^{-i}, S_i^- transform in fundamental representation of $\widetilde{su}(2)$ i.e. they are rotated only by $\tilde J^i{}_j$ and satisfy (2.14). All the generators except K^a and P^a satisfy the Hermiteant conjugation rules in (2.6), while K^a and P^a are taken to be anti-Hermitean: $(P^\pm)^\dagger=-P^\pm$, $(K^\pm)^\dagger=-K^\pm$. The light-cone basis for generators described above is used in the calculation of the Cartan forms in Appendix E. Making the substitutions $P^\pm\to\mathrm{i}P^\pm$, $K^\pm\to-\mathrm{i}K^\pm$ we obtain the basis used in Sections 2-4.

Appendix C Derivation of supercharges

Here we would like to demonstrate how the knowledge of kinematical charges and commutation relations of superalgebra allows one to get dynamical charges systematically. Consider, for example, the dynamical supercharges Q^{-i} whose most general form is

$$Q^{-i} = Q^{-i}(p^+, \partial_{p^+}, z, \partial_z, \theta, \lambda, \eta, \vartheta)\,, \tag{C.1}$$

where a dependence on S^3 is not shown explicitly. From $[P^+, Q^{-i}] = 0$ we get

$$Q^{-i} = {}^{(1)}Q^{-i}(p^+, z, \partial_z, \theta, \lambda, \eta, \vartheta)\,, \tag{C.2}$$

i.e. we learn that Q^{-i} does not depend on ∂_{p^+}. From $\{Q^{-i}, Q^{+j}\} = 0$ we get

$${}^{(1)}Q^{-i}(p^+, z, \partial_z, \theta, \lambda, \eta, \vartheta) = {}^{(2)}Q^{-i}(p^+, z, \partial_z, \theta, \eta, \vartheta)\,, \tag{C.3}$$

i.e. Q^{-i} does not depend on λ. The anticommutator $\{Q^{-i}, Q_j^+\} = 0$ tell us that Q^{-i} does not depend on θ, i.e.

$${}^{(2)}Q^{-i}(p^+, z, \partial_z, \theta, \eta, \vartheta) = {}^{(3)}Q^{-i}(p^+, z, \partial_z, \eta, \vartheta)\,. \tag{C.4}$$

From $[Q^{-i}, K^+] = S^{+i}$ we get

$${}^{(3)}Q^{-i} = \frac{1}{\sqrt{2}}\eta^i\partial_z + {}^{(4)}Q^{-i}(p^+, z, \eta, \vartheta)\,, \tag{C.5}$$

i.e.

$$Q^{-i} = \frac{1}{\sqrt{2}}\eta^i\partial_z + {}^{(4)}Q^{-i}(p^+, z, \eta, \vartheta)\ , \tag{C.6}$$

and from $\{Q^{-i}, S^{+j}\} = 0$ we get

$${}^{(4)}Q^{-i} = -\frac{1}{\sqrt{2}z}\eta^i(\eta\vartheta) + {}^{(5)}Q^{-i}(p^+, z, \eta) \tag{C.7}$$

i.e.

$$Q^{-i} = \frac{1}{\sqrt{2}}\eta^i\partial_z - \frac{1}{\sqrt{2}z}\eta^i(\eta\vartheta) + {}^{(5)}Q^{-i}(p^+, z, \eta)\ . \tag{C.8}$$

The second anticommutator in (2.12) gives

$${}^{(5)}Q^{-i}(p^+, z, \eta) = \frac{1}{2\sqrt{2}z}\eta^i + \frac{2}{\sqrt{2}z}(l\eta)^i + {}^{(6)}Q^{-i}(p^+, z)\ , \tag{C.9}$$

and the $su(2)$ covariance implies ${}^{(6)}Q^{-i}(p^+, z) = 0$. Taking this into account and plugging (C.9) into (C.8) we get Q_i^- given by (3.16). Using the Hermitean conjugation rule $(Q^{-i})^\dagger = Q_i^-$ we get the expression for Q_i^- in (3.15). The anticommutator $\{Q_i^-, Q^{-j}\} = -P^-\delta_i^j$ determines P^-, i.e. the Hamiltonian. The remaining dynamical generators K^-, S^{-i}, S_i^- can be obtained in a similar way.

Appendix D Eigenvectors of AdS mass operator

Here we would like to explain the procedure of finding the eigenvectors of the AdS mass operator A or the operator X in (3.19) in Section 4.1. Since the superfield $\Phi_{k,\sigma}$ in (4.11) diagonalizes the operators l^2 and $\eta\vartheta$ we have to diagonalize the operator $\vartheta l\eta \equiv \vartheta_i l^i{}_j \eta^j$, i.e. to find the solution to equation

$$\vartheta l\eta \ \Phi_{k,1} = \ m \ \Phi_{k,1} \ , \tag{D.1}$$

where m is an eigenvalue. We look for the following most general solution

$$\Phi_{k,1} = (\vartheta_i + c(\vartheta l)_i)\Phi^i_{k,1} \ , \tag{D.2}$$

where Φ^i does not depend on ϑ and c should be determined. Making use of

$$\{(\vartheta l)_j, (l\eta)^i\} = \frac{1}{2}(l^2 + 2\vartheta l\eta)\delta^i_j + (\eta\vartheta - 1)l^i{}_j \tag{D.3}$$

and $\eta^j\Phi^i_{k,1} = 0$ we get

$$(\vartheta l\eta)\Phi_{k,1} = \Big((1+c)(\vartheta l)_i + \frac{k(k+2)}{4}c\vartheta_i\Big)\Phi^i_{k,1} \,. \tag{D.4}$$

From (D.1) we find then the equations

$$\frac{k(k+2)}{4}c = m \,, \qquad (1+c) = mc \,, \tag{D.5}$$

which are solved by

$$m^{(1)} = -\frac{k}{2} \,, \qquad c^{(1)} = -\frac{2}{k+2} \,; \qquad m^{(2)} = \frac{k+2}{2} \,, \qquad c^{(2)} = -\frac{2}{k} \,. \tag{D.6}$$

Thus we have the two solution and the two eigenvectors

$$\Phi^{(1)}_{k,1} = (\vartheta_i - \frac{2}{k+2}(\vartheta l)_i)\Phi^i_{k,1}, \qquad \Phi^{(2)}_{k,1} = (\vartheta_i - \frac{2}{k}(\vartheta l)_i)\Phi^i_{k,1} \,. \tag{D.7}$$

Taking into account the relation

$$2l^2\Phi^{(1,2)}_{k,1} = k(k+2)\Phi^{(1,2)}_{k,1} \,, \qquad (\eta\vartheta - 1)\Phi^{(1,2)}_{k,1} = 0 \,, \tag{D.8}$$

and the eigenvalues $m^{(1)}$ and $m^{(2)}$ of $\vartheta l\eta$ given in (D.6) we get the following eigenvalues of the operator X

$$X\Phi^{(1)}_{k,1} = k^2\Phi^{(1)}_{k,1} \,, \qquad X\Phi^{(2)}_{k,1} = (k+2)^2\Phi^{(2)}_{k,1} \,. \tag{D.9}$$

Appendix E Superstring action

The standard kinetic term of superstring action in $AdS_3 \times S^3$ [14–16]

$$\mathcal{L}_{kin} = -\frac{1}{2}\sqrt{g}g^{\mu\nu}\Big(\hat{L}^A_\mu \hat{L}^A_\nu + L^{A'}_\mu L^{A'}_\nu\Big)\,, \tag{E.1}$$

can be rewritten in conformal algebra notation as [19]

$$\mathcal{L}_{kin} = -\frac{1}{2}\sqrt{g}g^{\mu\nu}\Big(\hat{L}^a_\mu \hat{L}^a_\nu + L_{D\mu}L_{D\nu} + L^{A'}_\mu L^{A'}_\nu\Big)\,, \tag{E.2}$$

where the Cartan 1-forms

$$\hat{L}^a \equiv L^a_{\scriptscriptstyle P} - \frac{1}{2}L^a_{\scriptscriptstyle K}\,, \qquad L^{A'} \equiv -\frac{\mathrm{i}}{2}(\sigma^{A'})^i{}_j L^j{}_i + \frac{\mathrm{i}}{2}(\sigma^{A'})^i{}_j \tilde{L}^j{}_i\,, \tag{E.3}$$

in the light-cone basis are defined by

$$\begin{aligned} G^{-1}dG &= L^a_{\scriptscriptstyle P}P^a + L^a_{\scriptscriptstyle K}K^a + L_{\scriptscriptstyle D}D + L^{-+}J^{+-} + L^i{}_j J^j{}_i + \tilde{L}^i{}_j \tilde{J}^j{}_i \\ &+ L^{-i}_{\scriptscriptstyle Q}Q^+_i + L^-_{\scriptscriptstyle Q\,i}Q^{+i} + L^{+i}_{\scriptscriptstyle Q}Q^-_i + L^+_{\scriptscriptstyle Q\,i}Q^{-i} + L^{-i}_{\scriptscriptstyle S}S^+_i + L^-_{\scriptscriptstyle S\,i}S^{+i} + L^{+i}_{\scriptscriptstyle S}S^-_i + L^+_{\scriptscriptstyle S\,i}S^{-i}\,, \end{aligned} \tag{E.4}$$

where the generators are taken in the basis described in Appendix B. To represent the Cartan 1-forms in terms of the even and odd coordinate fields we shall start with the following supercoset representative

$$\begin{aligned} G &= \exp(x^a P^a + \theta^{-i}Q^+_i + \theta^-_i Q^{+i} + \theta^{+i}Q^-_i + \theta^+_i Q^{-i}) \\ &\quad \times \exp(\eta^{-i}S^+_i + \eta^-_i S^{+i} + \eta^{+i}S^-_i + \eta^+_i S^{-i})\, g_y\ g_\phi\ , \end{aligned} \tag{E.5}$$

where g_ϕ and g_y depend on the radial AdS_3 coordinate ϕ and S^3 coordinates $y^{A'}$ respectively:

$$g_\phi \equiv \exp(\phi D)\ , \qquad g_y \equiv \exp(y^i{}_j(J^j{}_i - \tilde{J}^j{}_i))\,, \qquad y^i{}_j \equiv \frac{\mathrm{i}}{2}(\sigma^A)^i{}_j y^{A'}\ . \tag{E.6}$$

Choosing the parametrization of the coset representative in the form (E.5) corresponds to what is usually referred to as "Killing gauge" in superspace. Eq. (E.4) provides the definition of the Cartan forms in the light-cone basis. Let us further specify them by setting to zero some of the fermionic coordinates which corresponds to fixing a particular κ-symmetry gauge. Namely, we shall fix the κ-symmetry by putting to zero all the Grassmann coordinates which carry positive J^{+-} charge: $\theta^{+i} = \theta^+_i = \eta^{+i} = \eta^+_i = 0$. To simplify the notation in what follows we shall set $\theta^i \equiv \theta^{-i}$, $\theta_i \equiv \theta^-_i$, $\eta^i \equiv \eta^{-i}$, $\eta_i \equiv \eta^-_i$. Note that since S^{+i} and Q^{+i} transform in the fundamental representations of $su(2)$ and $\widetilde{su}(2)$ the corresponding fermionic coordinates η's and θ's also transform in the fundamental representation of $su(2)$ and $\widetilde{su}(2)$. As a result, the κ-symmetry fixed form of the coset representative (E.5) is

$$G_{g.f.} = \exp(x^a P^a + \theta^i Q^+_i + \theta_i Q^{+i})\ \exp(\eta^i S^+_i + \eta_i S^{+i})\ g_y\ g_\phi\ . \tag{E.7}$$

Plugging $G_{g.f.}$ into (E.4) we get the κ-symmetry gauge fixed expressions for the Cartan 1-forms

$$L_{P}^{+}=e^{\phi}dx^{+}\ ,\qquad L_{P}^{-}=e^{\phi}(dx^{-}-\frac{\mathrm{i}}{2}\tilde{\theta}^{i}\tilde{d\theta}_{i}-\frac{\mathrm{i}}{2}\tilde{\theta}_{i}\tilde{d\theta}^{i})\ , \tag{E.8}$$

$$L_{K}^{-}=e^{-\phi}[\frac{1}{4}(\tilde{\eta}^{2})^{2}dx^{+}+\frac{\mathrm{i}}{2}\tilde{\eta}^{i}\tilde{d\eta}_{i}+\frac{\mathrm{i}}{2}\tilde{\eta}_{i}\tilde{d\eta}^{i}]\ ,\qquad L_{D}=d\phi\ , \tag{E.9}$$

$$L^{i}{}_{j}=(dUU^{-1})^{i}{}_{j}+\mathrm{i}(\tilde{\eta}^{i}\tilde{\eta}_{j}-\frac{1}{2}\tilde{\eta}^{2}\delta^{i}_{j})dx^{+}\ ,\qquad \tilde{L}^{i}{}_{j}=(d\tilde{U}\tilde{U}^{-1})^{i}{}_{j}\ , \tag{E.10}$$

$$L_{Q}^{-i}=e^{\phi/2}\tilde{d\theta}^{i},\quad ,L_{Qi}^{-}=e^{\phi/2}\tilde{d\theta}_{i}\,,\quad L_{Q}^{+i}=-\mathrm{i}e^{\phi/2}\tilde{\eta}^{i}dx^{+},\quad L_{Qi}^{+}=\mathrm{i}e^{\phi/2}\tilde{\eta}_{i}dx^{+}, \tag{E.11}$$

$$L_{S}^{-i}=e^{-\phi/2}(\tilde{d\eta}^{i}+\frac{\mathrm{i}}{2}\tilde{\eta}^{2}\tilde{\eta}^{i}dx^{+})\ ,\qquad L_{Si}^{-}=e^{-\phi/2}(\tilde{d\eta}_{i}-\frac{\mathrm{i}}{2}\tilde{\eta}^{2}\tilde{\eta}_{i}dx^{+})\ , \tag{E.12}$$

with all the remaining forms equal to zero. We have introduced the notation

$$\tilde{\eta}^{i}\equiv U^{i}{}_{j}\eta^{j}\ ,\qquad \tilde{\eta}_{i}\equiv\eta_{j}(U^{-1})^{j}{}_{i}\ ,\qquad \tilde{\theta}^{i}\equiv\tilde{U}^{i}{}_{j}\theta^{j}\ ,\qquad \tilde{\theta}_{i}\equiv\theta_{j}(\tilde{U}^{-1})^{j}{}_{i}\ , \tag{E.13}$$

$$\widetilde{d\eta}^{i}\equiv U^{i}{}_{j}d\eta^{j}\ ,\qquad \widetilde{d\eta}_{i}\equiv d\eta_{j}(U^{-1})^{j}{}_{i}\ ,\qquad \widetilde{d\theta^{i}}\equiv\tilde{U}^{i}{}_{j}d\theta^{j}\ ,\qquad \widetilde{d\theta}_{i}\equiv d\theta_{j}(\tilde{U}^{-1})^{j}{}_{i}\ . \tag{E.14}$$

The fact that θ^{i} and η^{i} are rotated by $\widetilde{SU}(2)$ and $SU(2)$ is related to the presence of the matrices $\tilde{U}\in\widetilde{SU}(2)$ and $U\in SU(2)$ in the definition of $\tilde{\theta}^{i}$ and $\tilde{\eta}^{i}$. These matrices defined by (5.4) can be written explicitly as

$$U=\cos\frac{|y|}{2}+\mathrm{i}\sigma^{A'}n^{A'}\sin\frac{|y|}{2}\,,\qquad \tilde{U}=\cos\frac{|y|}{2}-\mathrm{i}\sigma^{A'}n^{A'}\sin\frac{|y|}{2}\ , \tag{E.15}$$

where $|y|$ and $n^{A'}$ are given by (A.13).

The S^{3} components $L^{A'}$ of the Cartan forms defined by (E.3) can be written in the following equivalent ways

$$L^{A'}=e^{A'}-\frac{1}{2}\tilde{\eta}_{i}(\sigma^{A'})^{i}{}_{j}\tilde{\eta}^{j}dx^{+}\quad ,\qquad L^{A'}=e^{A'}_{\mathcal{A}}(dy^{\mathcal{A}}+\mathrm{i}\eta_{i}(V^{\mathcal{A}})^{i}{}_{j}\eta^{j}dx^{+}) \tag{E.16}$$

where $e^{A'}_{\mathcal{A}}$ and $(V^{\mathcal{A}})^{i}{}_{j}$ are defined by (A.12) and (A.15) and we used the relation

$$e^{\mathcal{A}}_{A'}(U^{\dagger}\sigma^{A'}U)^{i}{}_{j}=-2\mathrm{i}(V^{\mathcal{A}})^{i}{}_{j}\,,\qquad e^{\mathcal{A}}_{A'}e^{\mathrm{B}'}_{\mathcal{A}}=\delta^{\mathrm{B}'}_{A'}\,. \tag{E.17}$$

Plugging the Cartan 1-forms into the kinetic part of the string Lagrangian (E.2) we get

$$\begin{aligned}\mathcal{L}_{kin}\ &=\ -\sqrt{g}g^{\mu\nu}\Big[e^{2\phi}\partial_{\mu}x^{+}\partial_{\nu}x^{-}+\frac{1}{2}\partial_{\mu}\phi\partial_{\nu}\phi+\frac{1}{2}L^{A'}_{\mu}L^{A'}_{\nu}\Big] \qquad\qquad (E.18)\\ &+\ \sqrt{g}g^{\mu\nu}\partial_{\mu}x^{+}\Big[(e^{2\phi}(\frac{\mathrm{i}}{2}\theta^{i}\partial_{\nu}\theta_{i}+\frac{\mathrm{i}}{2}\theta_{i}\partial_{\nu}\theta^{i})+\frac{\mathrm{i}}{4}\eta^{i}\partial_{\nu}\eta_{i}+\frac{\mathrm{i}}{4}\eta_{i}\partial_{\nu}\eta^{i}+\frac{1}{8}(\eta^{2})^{2}\partial_{\nu}x^{+}\Big]\,.\end{aligned}$$

In order to get the action in the Killing parametrization one needs to use $L^{A'}$ given by the second expression in (E.16) and then make the rescalings

$$\eta^{i}\to\sqrt{2}e^{\phi}\eta^{i},\quad \eta_{i}\to\sqrt{2}e^{\phi}\eta_{i}\,,\qquad x^{a}\to-x^{a}\,. \tag{E.19}$$

The action in the WZ parametrization is found by using $L^{A'}$ given by the first expression in (E.16) and after the transformation

$$\theta^i \to (\tilde{U}^{-1})^i{}_j\theta^i, \quad \theta_i \to \theta_j\tilde{U}^j{}_i\,, \quad \eta^i \to \sqrt{2}e^{\phi}(U^{-1})^i{}_j\eta^j, \quad \eta_i \to \sqrt{2}e^{\phi}\eta_jU^j{}_i\,, \quad x^a \to -x^a\,, \tag{E.20}$$

and use of the Fierz rearrangement rule $(\eta_i(\sigma^{A'})^i{}_j\eta^j)^2 = -3(\eta^2)^2$.

The P-odd WZ part of the covariant string Lagrangian $\mathcal{L}_{WZ}$ (see e.g. [14–16]) takes the following form in the light-cone gauge

$$\mathcal{L}_{WZ} = -\frac{\mathrm{i}}{\sqrt{2}}\epsilon^{\mu\nu}L^{+i}_{Q\mu}C'_{ij}L^{-j}_{Q\nu} + h.c.\,. \tag{E.21}$$

Plugging in the expressions for the Cartan 1-forms and making the rescalings given above we get $\mathcal{L}_{WZ}$ in the Killing parametrization (5.22) (see also (5.27)) and in the WZ parametrization (see the $\epsilon^{\mu\nu}$ term in (5.6)).

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